On the Equivalence of the Graph-Structural and Optimization-Based Characterizations of Popular Matchings

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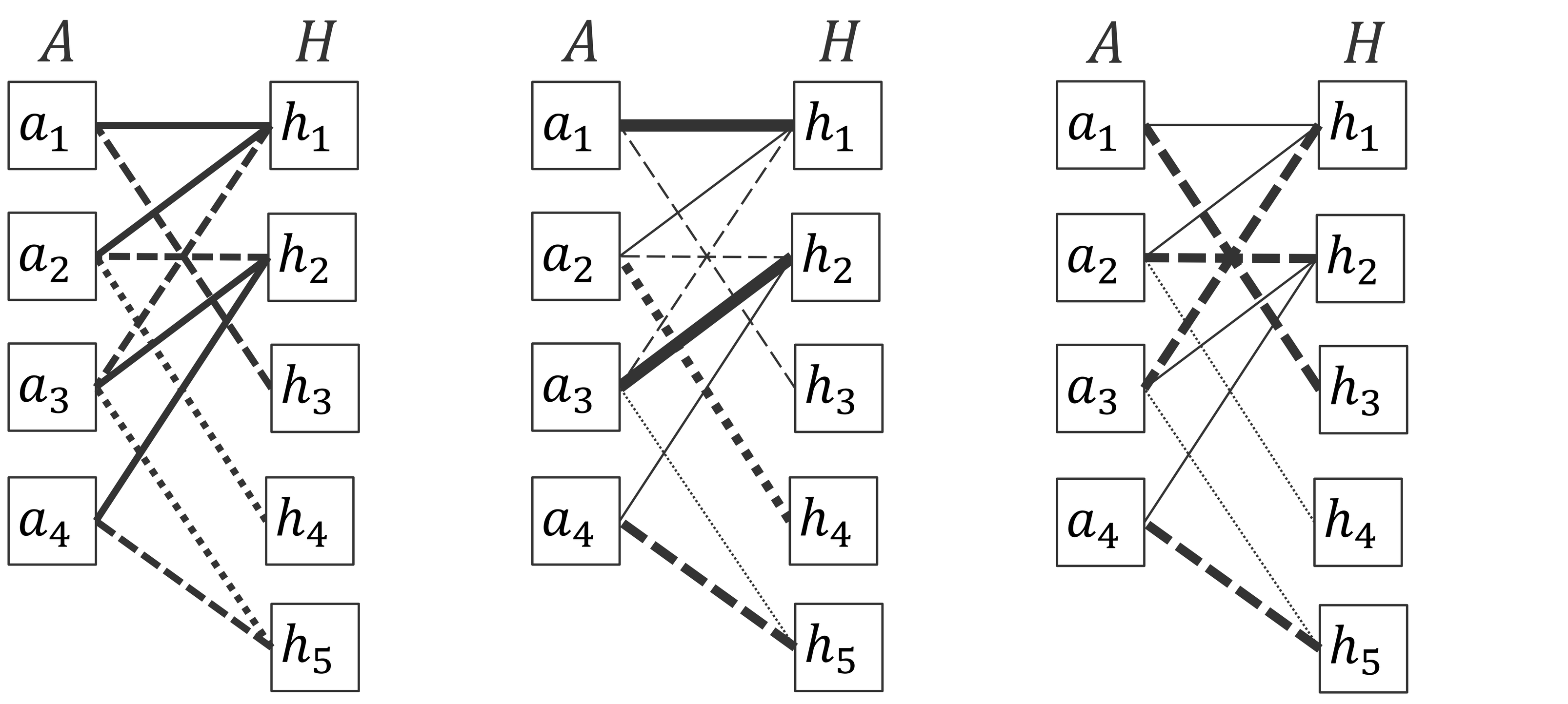
Abstract

In a bipartite graph in which the vertices have preferences over their neighbors, a **popular matching** is a matching which does not lose in a majority vote against any matching. In the literature, we have a **graph-structural characterization** and an **optimization-based characterization** described by maximum-weight matchings. **A main contribution of this paper is a direct connection of the two characterizations**, which suggests a new interpretation of the graph-structural characterization in terms of the dual optimal solution for the maximum-weight matching problem.

House Allocation model (HA model)

- $(A, H; E)$: Bipartite graph
- $\succ_a$: Preference of an applicant $a \in A$ over the houses in H
- $M, N \subseteq E$: Matchings in $(A, H; E)$
  - $M(a) \in H$: House matched to $a \in A$ by M
  - $\Delta(M, N) = \{a \in A \mid M(a) \succ_a N(a)\}$ ◂ a prefers M to N

Defn M is a **popular matching** if
$\Delta(M,N) - \Delta(N,M) \geq 0$ for every matching N



—— $\succ_{a_i}$ - - - - $\succ_{a_i}$ ⋯⋯ M_1: Popular M_2: Not popular

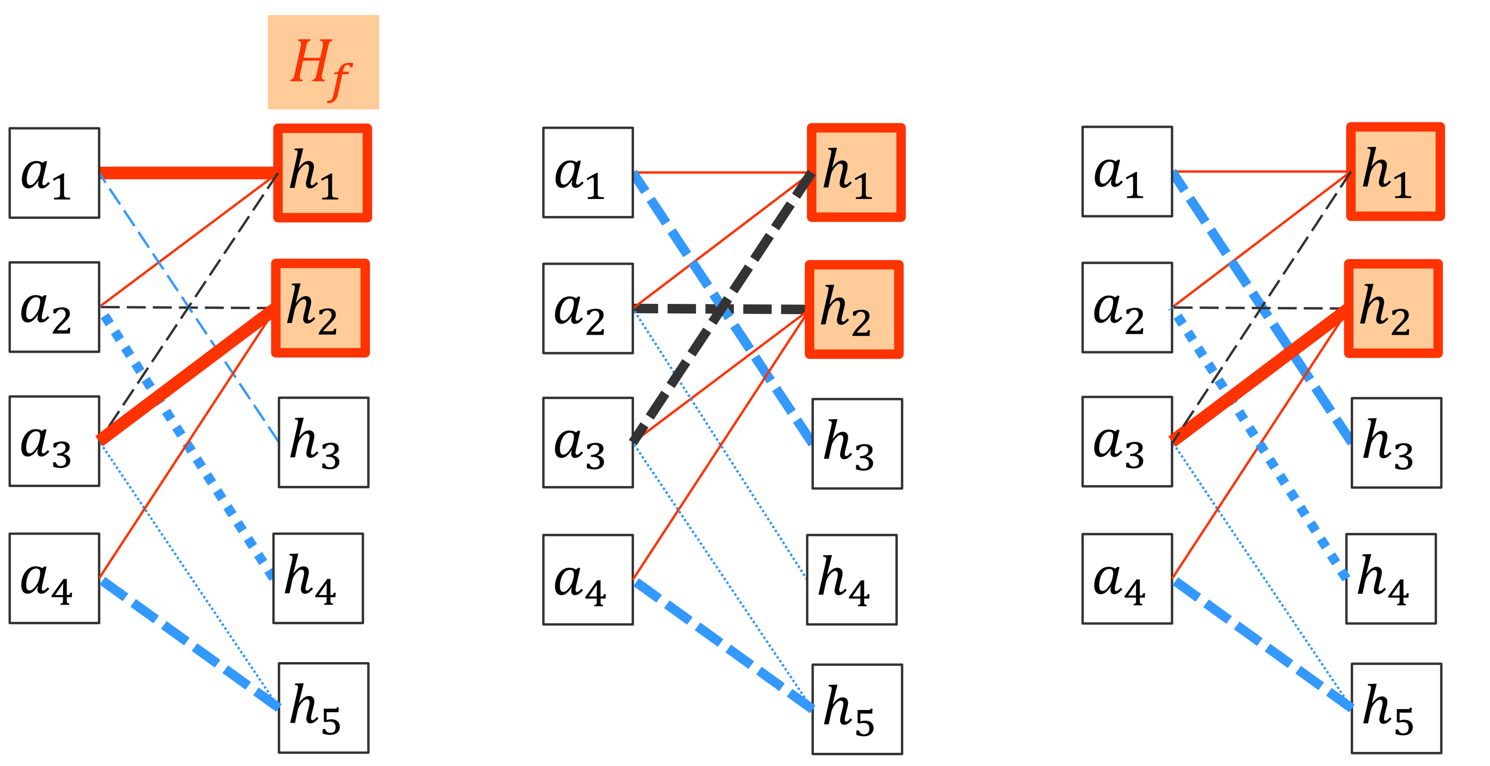
- M_2 is not popular, because $\Delta(M_2, M_1) - \Delta(M_1, M_2) < 0$

- **Then, how can we efficiently verify that M_1 is popular?**

Graph-Structural Characterization

- $f(a) \in H$: House most preferred by $a \in A$
- $H_f = \bigcup_{a \in A}\{f(a)\} = \{h \mid \exists a \in A, h = f(a)\}$
- $s(a) \in H$: House in $H \setminus H_f$ most preferred by $a \in A$

Theorem [Abraham, Irving, Kavitha, Mehlhorn, 2007]
A matching M is popular if and only if
(GS1) Each $h \in H_f$ is matched by M, and
(GS2) Each $a \in A$ is matched to $f(a)$ or $s(a)$ by M



M_1 satisfies **GS1,2** M_2 violates **GS2** M_3 violates **GS1**

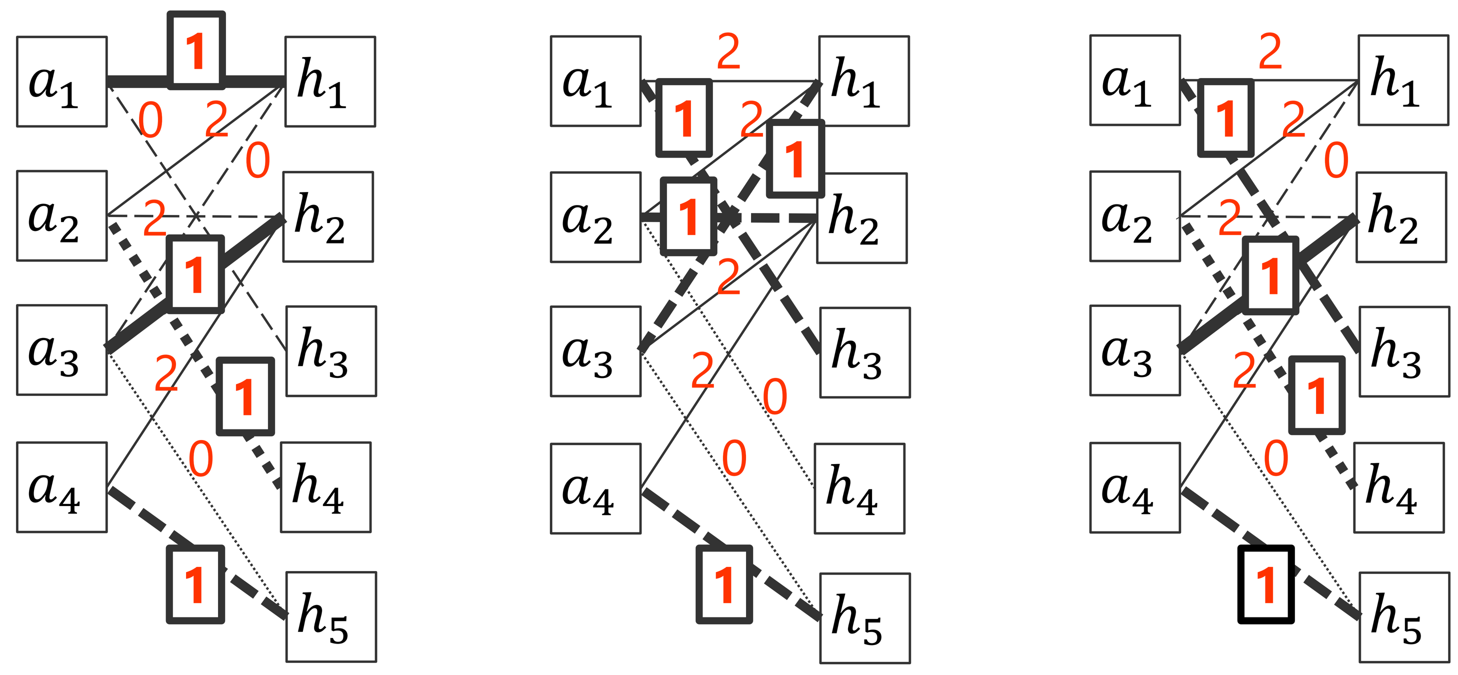
- For simplicity, we add a last resort (least preferred house) $\ell(a)$ for each $a \in A$, to assume that each $a \in A$ is matched

Optimization-Based Characterization

- Given a matching M, define edge weights $w_M \in \{0,1,2\}^E$ by

$$w_M(a,h) = \begin{cases} 2 & \text{if } h \succ_a M(a) \\ 1 & \text{if } h = M(a) \\ 0 & \text{if } h \prec_a M(a) \end{cases}$$

Theorem [Biró, Irving, Manlove, 2010]
A matching M is popular if and only if
(Opt) M is a **max-weight matching wrt** w_M



M_1: **Max-weight** M_2: **Not** max-weight M_3: **Not** max-weight

Our Contribution
Direct proof of (GS1)(GS2) ↔ (Opt)

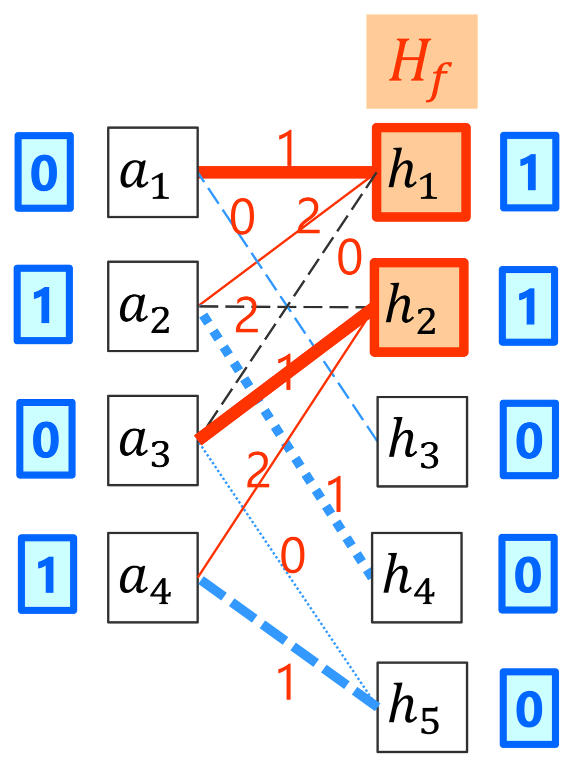
Proof Sketch of (GS1)(GS2)→(OPT)

- **(Dual)** Minimize $\sum_{a \in A} y(a) + \sum_{h \in H} y(h)$
  subject to $y(a) + y(h) \geq w_M(a,h) \quad \forall (a,h) \in E$
  $y(h) \geq 0 \quad \forall h \in H$
- Define $y \in \mathbb{R}^{A \cup H}$ by

$$y(a) = \begin{cases} 0 & \text{if } M(a) = f(a) \\ 1 & \text{if } M(a) = s(a) \end{cases}$$

$$y(h) = \begin{cases} 1 & \text{if } h \in H_f \\ 0 & \text{if } h \in H \setminus H_f \end{cases}$$

- $\sum_{a \in A} y(a) + \sum_{h \in H} y(h) = |A|$, and hence y is an optimal solution of **(Dual)** □



Corollary
Yet $w_M \in \{0,1,2\}^E$, **(Dual)** has a **$\{0,1\}$-optimal solution**

Further Contribution
The same result for
- **House Allocation model with Ties** (**HAT**)
- **Stable Matching model** (**SMI**, two-sided preferences)
  - GS characterization due to [Huang, Kavitha 2013]

References

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