A Unified Model of Congestion Games with Priorities

Two-sided Markets with Ties, Finite and Non-affine Delay Functions, and Pure Nash Equilibria

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Overview

- **Congestion Game**
- **Open Question We Solved**

Congestion Game [Rosenthal 1973]

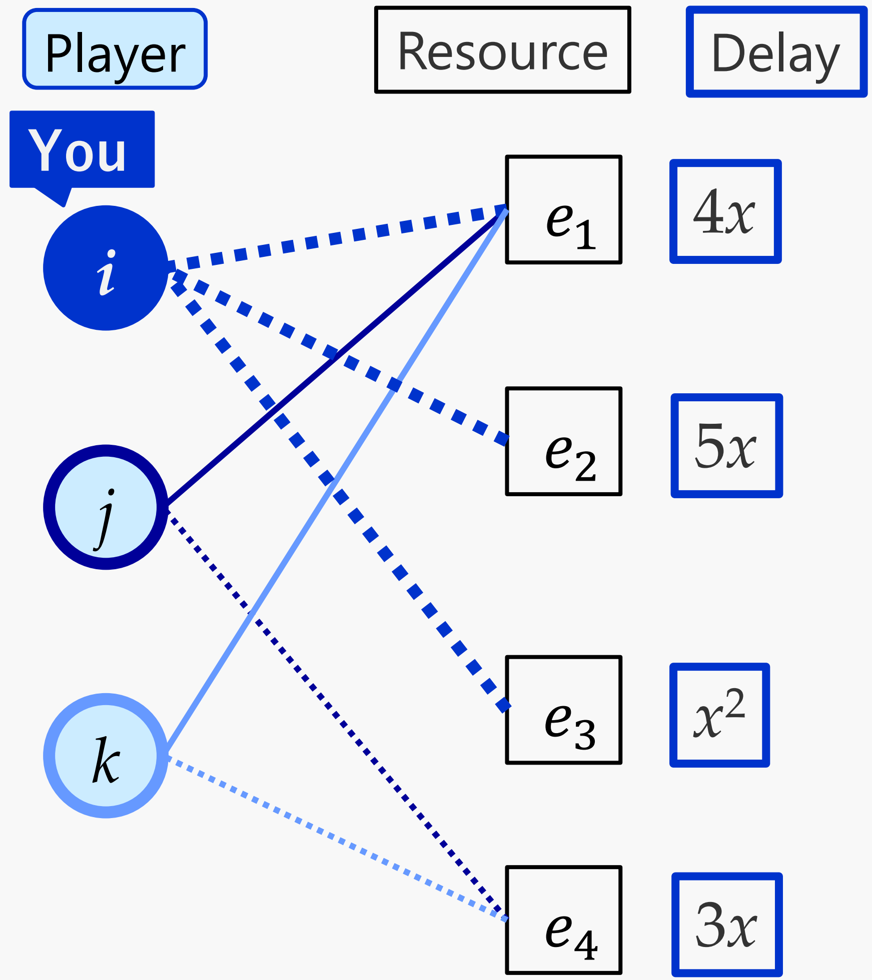
A model of **non-cooperative games**

Delay on the number x of players

- $e_1 : 4 \cdot 3 = 12$
- $e_2 : 5 \cdot 1 = 5$
- $e_3 : 1^2 = 1$

Pure Nash equilibrium (PNE)
No player can decrease her delay by only changing her resource

- A **standard model** in the analysis of PNE in non-cooperative games
- Also known as **routing game** or **selfish routing**



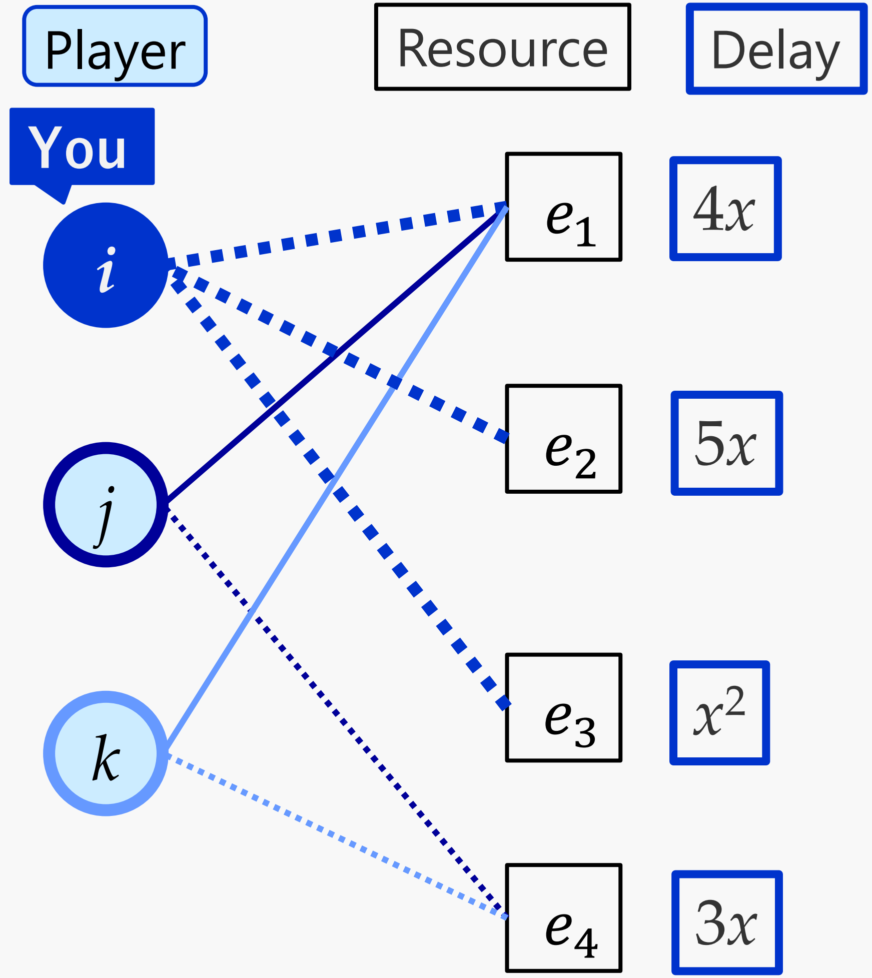
Congestion Game with Priorities

[Ackermann, Goldberg, Mirrokni, Röglin, Vöcking, 2008]

A common generalization
- **Congestion games**
- **Two-sided markets**

Resources have **priorities** over the players
- e_1: $i \sim j \prec k$ Preferring i and j to k

Delay imposed by e_1
- $\boldsymbol{i}$: $\mathbf{4 \cdot 2 = 8}$
- $\boldsymbol{j}$: $\mathbf{4 \cdot 2 = 8}$
- $\boldsymbol{k}$: $+\infty$



Open Question by Ackermann et al.

Recap

Less preferred players receive
an infinite delay
caused by more preferred players

Priorities

- $e_1 : i \sim j \prec k$ Preferring i and j to k

Delay by e_1 with delay func. $4x$

- i: $4 \cdot 2 = 8$
- j: $4 \cdot 2 = 8$
- k: $+\infty$

Open Question

How to design a model in which
less preferred players receive
a large but finite delay
caused by more preferred players ?

Our Solution

- **Designing such a model** as a common generalization of another model
- **Extending previous theorems on PNE**

Previous Work

- Congestion Game with Priorities
- Priority-Based Affine Congestion Game

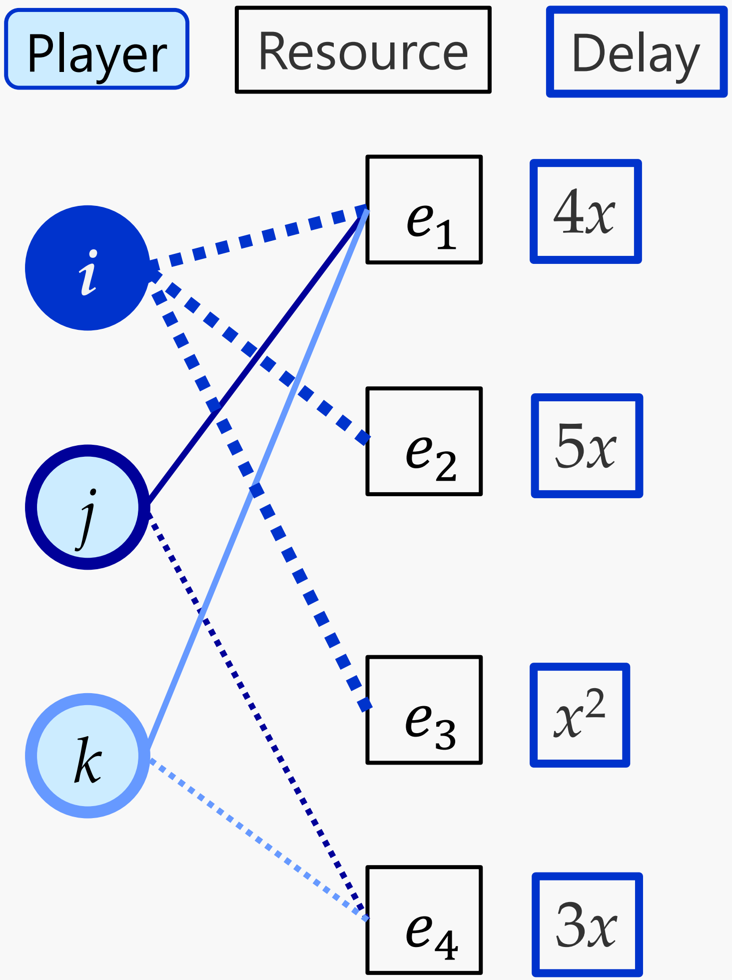
Congestion Game (Formal)

$G = (N, E, (\mathcal{S}_i)_{i \in N}, (d_e)_{e \in E})$

- N: **Players**
- E: **Resources**
- $\mathcal{S}_i$: **Strategy space** of player $i \in N$
  - $S_i \in \mathcal{S}_i$: **Strategy** of player $i \in N$
- $\boldsymbol{S} = (S_1, \dots, S_n)$: **State**
  - $N_e(\boldsymbol{S})$: **Players choosing** e
  - $n_e(\boldsymbol{S}) = |N_e(\boldsymbol{S})|$
- d_e: **Delay function** of resource $e \in E$

Delay on $i \in N$ in a state $\boldsymbol{S}$

$$\gamma_i(\boldsymbol{S}) = \sum_{e \in S_i} d_e(n_e(\boldsymbol{S}))$$



Classical Theorems on Congestion Games

Theorem [Rosenthal 1973]

Every congestion game admits an **exact potential function**

➢ Possesses a **PNE**

- **Exact potential function** Φ defined on the set of states

$$\Phi(\boldsymbol{S}_{-i}, S_i') - \Phi(\boldsymbol{S}) = \gamma_i(\boldsymbol{S}_{-i}, S_i') - \gamma_i(\boldsymbol{S}) \text{ for each } i \in N, \boldsymbol{S}, S_i' \in \mathcal{S}_i$$

- A state minimizing the potential Φ is a PNE

- Suffices to define Φ by

$$\Phi(\boldsymbol{S}) = \sum_{e \in E} \sum_{\ell=1}^{n_e(\boldsymbol{S})} d_e(\ell)$$

Theorem [Monderer, Shapley 1996]

Every exact potential game is a congestion game.

Congestion Game with Priorities (Formal)

$G = (N, E, (\mathcal{S}_i)_{i \in N}, (d_e)_{e \in E}, (p_e)_{e \in E})$

$p_e: N \to \mathbb{Z}$: **Priority function** of resource e

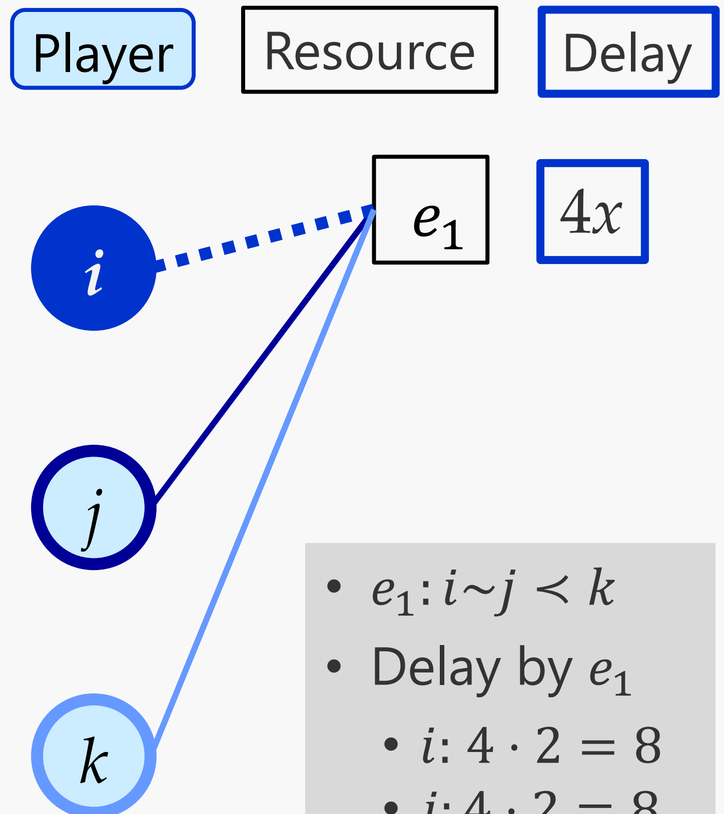
- $p_e(i) < p_e(j)$: e prefers player i to j

For a state $\boldsymbol{S}$ and a resource e,

- $p_e^*(\boldsymbol{S}) = \min\{p_e(i): i \in N_e(\boldsymbol{S})\}$
- $n_e^{p_e^*(\boldsymbol{S})} = |\{i \in N_e(\boldsymbol{S}): p_e(i) = p_e^*(\boldsymbol{S})\}|$

Delay on $i \in N_e(\boldsymbol{S})$ by e is

- $d_e(n_e^{p_e^*(\boldsymbol{S})})$ if $p_e(i) = p_e^*(\boldsymbol{S})$
- $+\infty$ if $p_e(i) > p_e^*(\boldsymbol{S})$



- $e_1: i \sim j \prec k$
- Delay by e_1
  - i: $4 \cdot 2 = 8$
  - j: $4 \cdot 2 = 8$
  - k: $+\infty$

Results by [Ackermann et al. 2008]

Theorem [Ackermann et al. 2008]

A **singleton** congestion game with priorities

- is a **potential game**,
- and hence possess a **PNE**

Each strategy is a singleton

More results by [Ackermann et al. 2008]

- Singleton, **identical priority function** $p_e(\cdot)$
  ➔ **PNE** is attained by poly. number of **better-response dynamics**
- Singleton, **player-specific delay function** $d_e(\cdot)$
  ➔ **PNE** can be computed in **poly. time**
- Extension from singleton game to **matroid game**

Smaller class

Larger class

Strategy space = Base family

Q. Can we get rid of the **infinite delay** ?

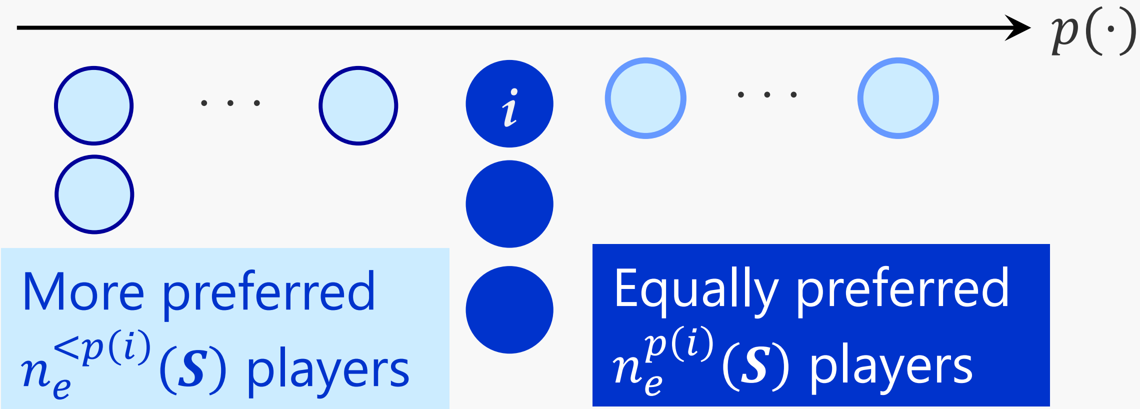
Delay on i is $+\infty$ if $p_e(i) > p_e^*(\boldsymbol{S})$

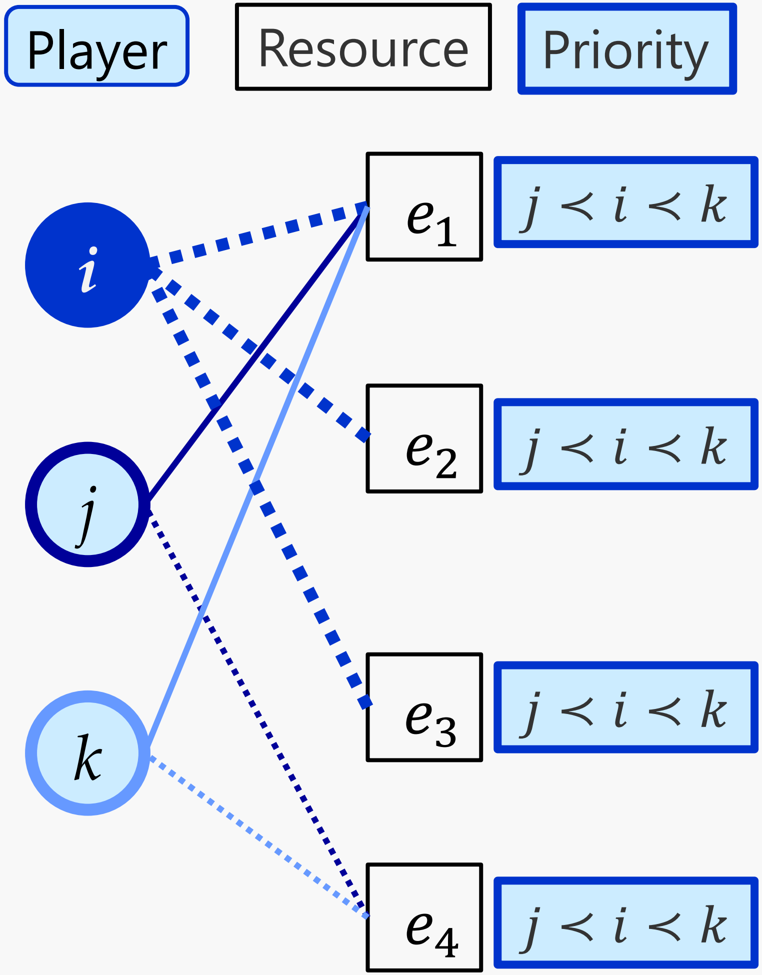
Priority-Based Affine Congestion Game [Bilò, Vinci 2023]

$G = (N, E, (\mathcal{S}_i)_{i \in N}, p, (\alpha_e, \beta_e)_{e \in E})$

- p: Priority function of **all resources**
- $(\alpha_e, \beta_e) \in \mathbb{R}^2$ defines the delay func. of e
  - $n_e^{<p(i)}(\boldsymbol{S}) = |\{j \in N_e(\boldsymbol{S}) : p(j) < p(i)\}|$
  - Delay on $i \in N_e(\boldsymbol{S})$ by e is

$$\alpha_e \cdot \left(n_e^{<p(i)}(\boldsymbol{S}) + \frac{n_e^{p(i)}(\boldsymbol{S}) + 1}{2} \right) + \beta_e$$





Results by [Bilò and Vinci 2023]

Theorem [Bilò and Vinci 2023]

A priority-based affine congestion game possesses a **PNE**

More results on **Price of Anarchy** and **Price of Stability**

Difference from [Ackermann et al. 2008]

- All resources have **the identical priority function** $p(\cdot)$
- A specific kind of **affine delay function**:

$$\alpha_e \cdot \left(n_e^{<p(i)}(\boldsymbol{S}) + \frac{n_e^{p(i)}(\boldsymbol{S})+1}{2} \right) + \beta_e$$

Q. Can we get rid of the **identical priorities** and **affine delay functions**?

Our Model

- Congestion Game with Priorities
- Priority-Based Affine Congestion Game
- **Priority-Based Congestion Game**

Our Model: Priority-Based Congestion Game

$G = (N, E, (\mathcal{S}_i)_{i \in N}, (d_e)_{e \in E}, (p_e)_{e \in E})$

Differences from the previous models

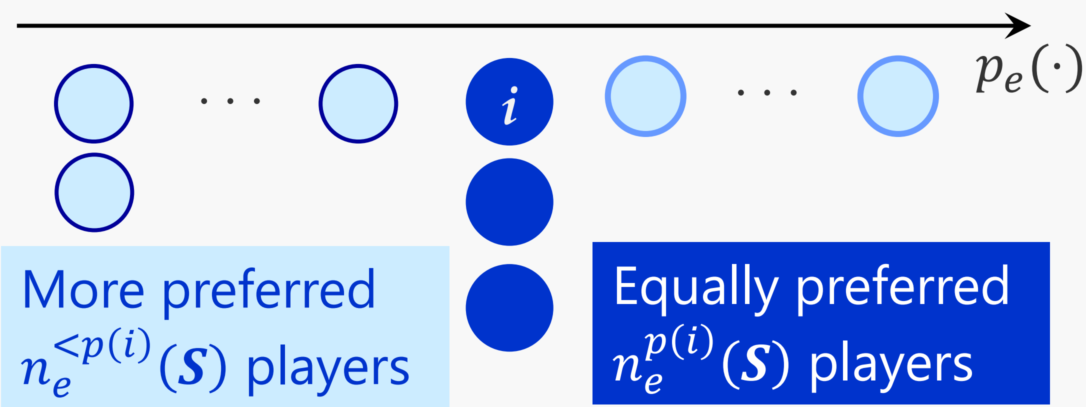
- $d_e \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$: **Bivariate delay function**
  - The delay on i by e is $d_e\left(n_e^{<p_e(i)}(\boldsymbol{S}), n_e^{p_e(i)}(\boldsymbol{S})\right)$

[Ackermann et al. 2008]

- $d_e(n_e^{p_e^*(\boldsymbol{S})})$ if $p_e(i) = p_e^*(\boldsymbol{S})$
- $+\infty$ if $p_e(i) > p_e^*(\boldsymbol{S})$

[Bilò and Vinci 2023]

$$\alpha_e \cdot \left(n_e^{<p(i)}(\boldsymbol{S}) + \frac{n_e^{p(i)}(\boldsymbol{S}) + 1}{2}\right) + \beta_e$$



The delay func. of [Ackermann et al. 2008] is obtained by defining $d'_e(x, y)$ by

- $d_e(y)$ if $x = 0$
- $+\infty$ if $x \geq 1$

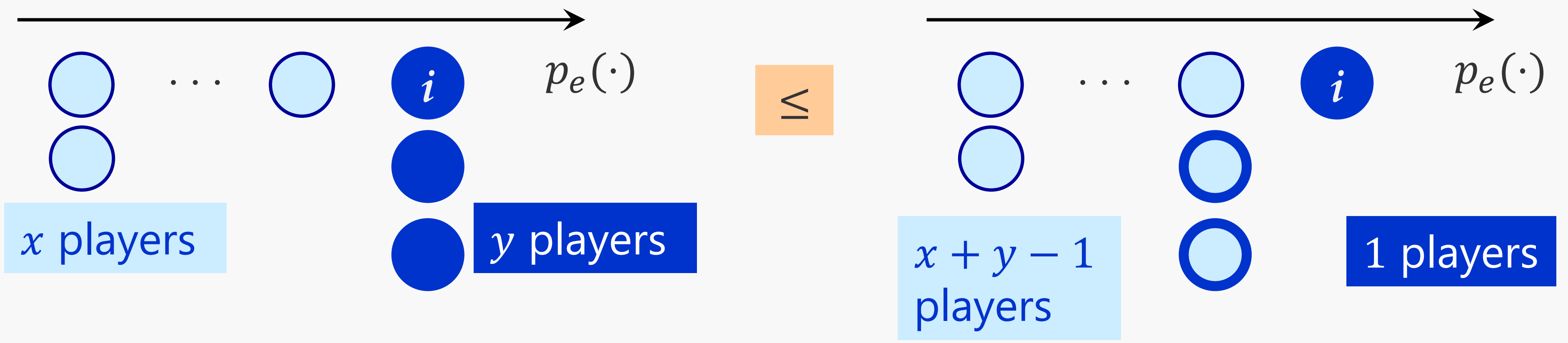
Assumption on the Delay Functions

Our result

Extension of the theorems of [Ackermann et al. 2008] to our model under an assumption on the delay functions

Assumption on the delay functions $d_e : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$:

1. $d_e(x, y) \leq d_e(x', y)$ if $x < x'$
2. $d_e(x, y) \leq d_e(x, y')$ if $y < y'$
3. $d_e(x, y) \leq d_e(x + y - 1, 1)$



Theorems on PNE

[Ackermann et al.]	Identical Priorities	General Priorities
Non-player-specific Delay	**Poly. better-response dynamics** • Singleton game • Matroid game	**Potential game** • Singleton game • Matroid game
Player-specific Delay	---	**Polynomial Algorithm** • Singleton game • Matroid game

Our Results	**Identical Priorities**	**General Priorities**
Non-player-specific Delay	**Poly. better-response dynamics** • Singleton game • Matroid game	**Potential game** • Singleton game • Matroid game
Player-specific Delay	**Poly. better-response dynamics** • Singleton game • Matroid game	**Existence of PNE** • Singleton game • Matroid game

One Proof

Theorem A priority-based singleton congestion game is a **potential game**, and hence possess a **PNE**

Proof Define a **potential** $\Phi(\boldsymbol{S}) \in (\mathbb{R} \times \mathbb{Z})^n$ of $\boldsymbol{S} = (e_1, \dots, e_n)$

- Resource e contributes the following $n_e(\boldsymbol{S})$ vectors in $\mathbb{R} \times \mathbb{Z}$
  - $(d_e(0,1), q_1), (d_e(0,2), q_1), \dots, \left(d_e\left(0, n_e^{q_1}(\boldsymbol{S})\right), q_1\right), \dots,$
  - $\left(d_e\left(n_e^{<q_k}(\boldsymbol{S}), 1\right), q_k\right), \left(d_e\left(n_e^{<q_k}(\boldsymbol{S}), 2\right), q_k\right), \dots, \left(d_e\left(n_e^{<q_k}(\boldsymbol{S}), n_e^{q_1}(\boldsymbol{S})\right), q_k\right), \dots,$
  - $\left(d_e\left(n_e^{<q_\ell}(\boldsymbol{S}), 1\right), q_\ell\right), \left(d_e\left(n_e^{<q_\ell}(\boldsymbol{S}), 2\right), q_\ell\right), \dots, \left(d_e\left(n_e^{<q_\ell}(\boldsymbol{S}), n_e^{q_\ell}(\boldsymbol{S})\right), q_\ell\right),$
  - where $q_1 < q_2 < \cdots < q_\ell$ is the priority values of the players in $N_e(\boldsymbol{S})$
- A **better response** from e to f of i **lexicographically decreases** $\Phi(\boldsymbol{S})$
  - f newly contributes $\left(d_f\left(n_f^{<p_f(i)}(\boldsymbol{S}), n_f^{p_f(i)}(\boldsymbol{S}) + 1\right), p_f(i)\right)$
  - This is **lex. smaller** than those disappeared, due to the fact that it is a better response and d_f satisfies Assumptions 1-3.

Conclusion

Summary and Future Work

Our contribution

A new model of congestion games with priorities

- Common generalization of the models of [Ackermann et al. 2008] and [Bilò and Vinci 2023]
- **Solution to the open question** of [Ackermann et al. 2008]
- Extending **the theorems on PNE**

Future work

- **Complexity analysis** of computing a PNE
- Analysis on **Price of Anarchy** and **Price of Stability**
- Weakening the assumption of the delay functions