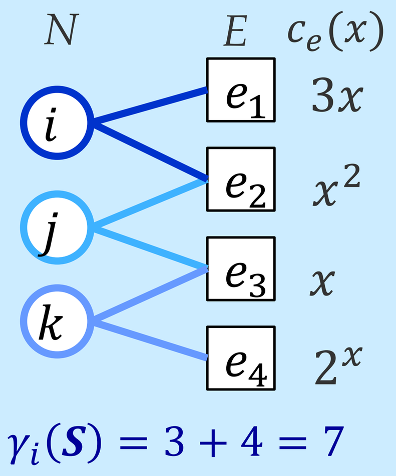
Pure Nash Equilibria in Weighted Congestion Games with Complementarities and Beyond

Kenjiro Takazawa Hosei University, Tokyo, Japan

Abstract
Congestion games offer a primary model of non-cooperative games, and a number of generalizations are in the literature, such as weighted congestion games and congestion games with complementarities. In this presentation, we prove the existence of pure Nash equilibria in weighted matroid congestion games with complementarities and their further generalization, under a simplified assumption.

Congestion Game

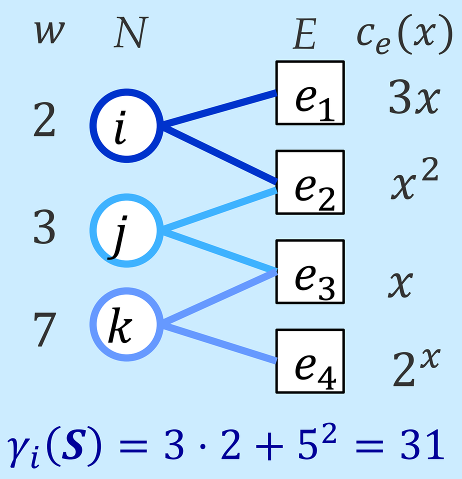
- N: Set of the players
- E: Set of the resources
- $\mathcal{S}_i \subseteq 2^E$: Strategy space of a player $i \in N$
- $c_e\colon \mathbb{Z} \to \mathbb{R}$: Cost function of a resource $e \in E$



- In a state $\boldsymbol{S} = (S_1, S_2, \dots, S_n)$,
  - $N_e(\boldsymbol{S}) = \{i \in N : e \in S_i\}$: Set of the players using $e \in E$
  - $n_e(\boldsymbol{S}) = |N_e(\boldsymbol{S})|$: Number of the players using $e \in E$
  - $c_e(n_e(\boldsymbol{S}))$: Cost of using a resource $e \in E$
  - $\gamma_i(\boldsymbol{S}) = \sum_{e \in S_i} c_e(n_e(\boldsymbol{S}))$: Cost imposed on a player $i \in N$

Theorem [Rosenthal '73]
Every congestion game has a **pure Nash equilibrium** (**PNE**)

Generelized Models



(A) Weighted Congestion Game
- $w_i \in \mathbb{R}$: Weight of a player $i \in N$
- The cost of $e \in E$ is $c_e\left(\sum_{i \in N_e(\boldsymbol{S})} w_i\right)$

(B) Congestion Game with Complementarities
- $f\colon \mathbb{R}^r \to \mathbb{R}$: **Aggregation function**
- The cost on $i \in N$ with strategy $S_i = (e_1, e_2, \dots, e_r)$ is

$$\gamma_i(\boldsymbol{S}) = f\left(c_{e_1}\left(n_{e_1}(\boldsymbol{S})\right), c_{e_2}\left(n_{e_2}(\boldsymbol{S})\right), \dots, c_{e_r}\left(n_{e_r}(\boldsymbol{S})\right)\right)$$

  - **Bottleneck congestion game** $\gamma_i(\boldsymbol{S}) = \max\{3, 4\} = 4$
  - **L_2-norm aggregation** $\gamma_i(\boldsymbol{S}) = \sqrt{3^2 + 4^2} = 5$

- Hereafter, the strategy space $\mathcal{S}_i \subseteq 2^E$ of each player $i \in N$ is the **base family of a matroid** (**Matroid congestion game**)
  - $S, S' \in \mathcal{S}_i, e \in S' - S \Rightarrow \exists f \in S' - s, S - e + e' \in \mathcal{S}_i$
  - **Better response** ▶ **Local better response**

Theorem [Ackermann, Röglin, Vöcking '09]
Every **weighted matroid congestion game** has a **PNE**

Theorem [Feldotto, Leder, Skopalik '17]
Every **matroid congestion game with complementarities** has a **PNE** if the aggregation function f is **weakly monotone**.

- **Weak monotonicity** of the aggregation function $f\colon \mathbb{R}^r \to \mathbb{R}$ is defined **in a very sophisticated manner !!**

Our Purpose
- **Common generalization** of **(A)** and **(B)**
- **Simpler weak monotonicity** ensuring PNE

Our Model
- N: Set of the players
- E: Set of the resources
- $\mathcal{S}_i \subseteq 2^E$: Strategy space of a player $i \in N$
- $c_e\colon 2^N \to \mathbb{R}$: Cost **set-function** of $e \in E$ [Takazawa '19]

- In a state $\boldsymbol{S} = (S_1, S_2, \dots, S_n)$,
  - $N_e(\boldsymbol{S}) = \{i \in N : e \in S_i\}$: Set of the players using $e \in E$
  - The cost of using a resource $e \in E$ is $c_e(N_e(\boldsymbol{S}))$
  - The cost on $i \in N$ with strategy $S_i = (e_1, e_2, \dots, e_r)$ is

$$\gamma_i(\boldsymbol{S}) = f\left(c_{e_1}\left(N_{e_1}(\boldsymbol{S})\right), c_{e_2}\left(N_{e_2}(\boldsymbol{S})\right), \dots, c_{e_r}\left(N_{e_r}(\boldsymbol{S})\right)\right)$$

Definition
- A function $f\colon \mathbb{R}^r \to \mathbb{R}$ is **weakly monotone** if (i) or (ii) holds for each $x, y \in \mathbb{R}$:
  (i) $f(\boldsymbol{v}, x) \leq f(\boldsymbol{v}, y)$ for each $\boldsymbol{v} \in \mathbb{R}^{r-1}$
  (ii) $f(\boldsymbol{v}, x) \geq f(\boldsymbol{v}, y)$ for each $\boldsymbol{v} \in \mathbb{R}^{r-1}$
  ▶ f determines a weak order $\precsim_f$ on $\mathbb{R}$
- A set function $c\colon 2^N \to \mathbb{R}$ is **monotonically nondecreasing with respect to f** if it holds that
  $f(\boldsymbol{v}, c(X)) \leq f(\boldsymbol{v}, c(Y))$ for each $\boldsymbol{v} \in \mathbb{R}^{r-1}$ and $X \subseteq Y \subseteq N$

Main Theorem
Every **matroid congestion game with set-functional costs and complementarities** has a **PNE** if f is **weakly monotone** and c_e is **monotonically nondecreasing wrt. f** for each resource e.

Proof Sketch
- $\boldsymbol{S}$: State, $i \in N$ has a better response
- **Matroid game** ▶ $\exists$ **Local better response** $S_i' = S_i - e + e'$
  - $c_{e'}\left(N_{e'}(\boldsymbol{S}')\right) \prec_f c_e(N_e(\boldsymbol{S}))$ ◀ Monotonicity of f
  - $c_e(N_e(\boldsymbol{S}')) \precsim_f c_e(N_e(\boldsymbol{S}))$ ◀ Monotonicity of c_e

  $\Rightarrow \left(c_e(\boldsymbol{S}')\right)_{e \in E} \precsim_{\mathrm{lex}} \left(c_e(\boldsymbol{S})\right)_{e \in E}$
  - If $c_e(N_e(\boldsymbol{S}')) \sim_f c_e(N_e(\boldsymbol{S}))$, then use $n_e(\boldsymbol{S}') < n_e(\boldsymbol{S})$

  $\Rightarrow (c_e(\boldsymbol{S}'), n_e(\boldsymbol{S}'))_{e \in E} \prec_{\mathrm{lex}} (c_e(\boldsymbol{S}), n_e(\boldsymbol{S}))_{e \in E}$
- The potential $(c_e(\boldsymbol{S}), n_e(\boldsymbol{S}))_{e \in E}$ strictly decreases ■

Corollary
A **weighted matroid bottleneck congestion game** has a **PNE** if the strategies of all players have the same size.

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