Notes on **Equitable Partitions** into **Matching Forests** in Mixed Graphs and ***b*-branchings** in Digraphs

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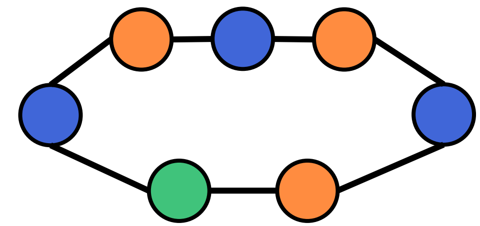
Equitable Coloring

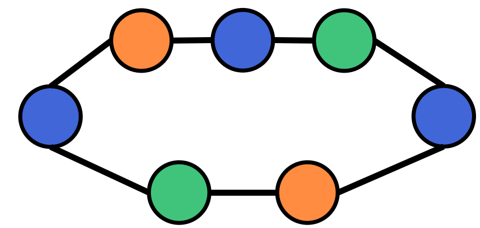
[Hajnal, Szemerédi '69] (Conjecture by **P. Erdős '64**)

Graph (V,E) with maximum degree Δ
➔ V can be partitioned into
Δ+1 stable sets $S_1,S_2,\ldots,S_{\Delta+1}$ such that
$\big||S_i| - |S_j|\big| \leq 1 \quad \forall i,j \in \{1,2,\ldots,\Delta+1\}$

➢ Namely, $|S_i| = \left\lfloor \frac{|V|}{\Delta+1} \right\rfloor$ or $\left\lceil \frac{|V|}{\Delta+1} \right\rceil$



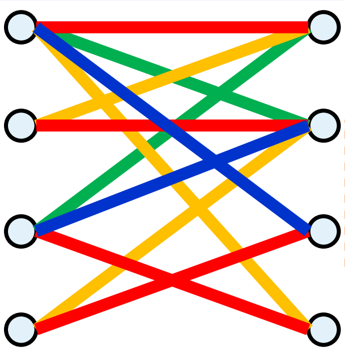
(3, 3, 1)
➔ ***Not equitable***



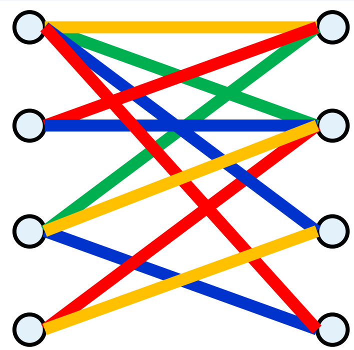
(3, 2, 2)
➔ ***Equitable***

[Folkman, Fulkerson '67] etc.

Bipartite graph (V,E) with maximum degree Δ
➔ For any $k \geq \Delta$, E can be partitioned into
***k* matchings** $M_1,M_2,\ldots,M_k$ such that
$\big||M_i| - |M_j|\big| \leq 1 \quad \forall i,j \in \{1,2,\ldots,k\}$



(4, 3, 2, 2)
➔ ***Not equitable***



(3, 3, 3, 2)
➔ ***Equitable***

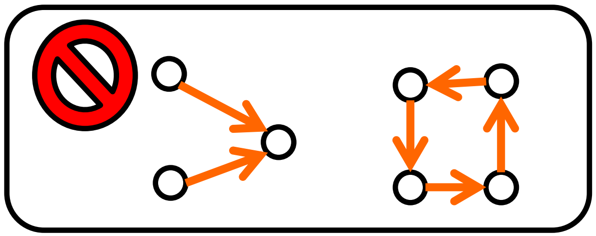
Equitable Partition into Branchings

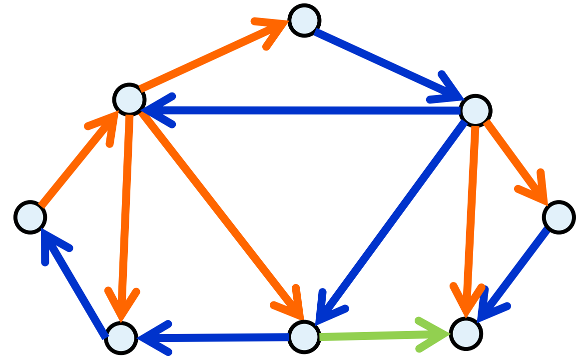
■ Digraph (V, A)

Definition

$B \subseteq A$ is a **branching** if

- (i) $\mathrm{indeg}(v) \leq 1 \quad \forall v \in V$
- (ii) No (undirected) cycle

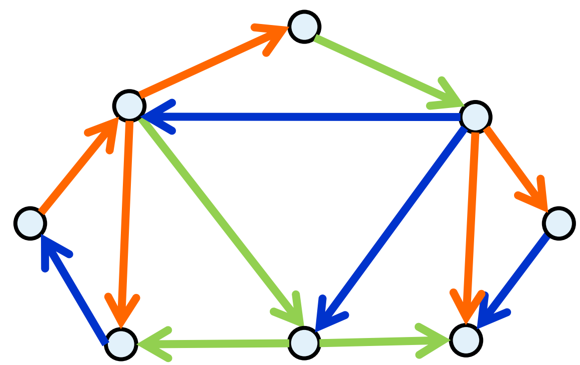




(6, 5, 1)
➔ ***Not equitable***

Theorem **[Schrijver 03]?**

If A can be partitioned into k branchings
➔ A can be partitioned into
k **branchings** $B_1, B_2, \ldots, B_k$ such that
$\boldsymbol{||B_i| - |B_j|| \leq 1} \quad \forall i, j \in [k]$

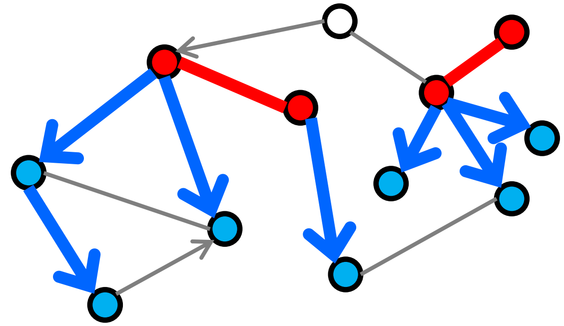


(4, 5, 4)
➔ ***Equitable***

Overview of Our Results

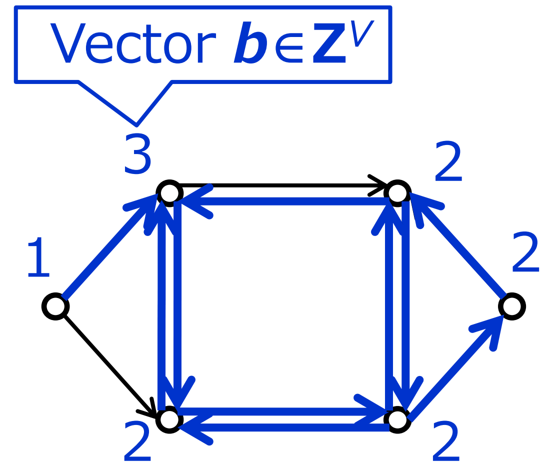
Theorem

For a **mixed graph** $(V, E\cup A)$,
if $E\cup A$ can be partitioned into k **matching forests**
➔ $E\cup A$ can be partitioned into
k **matching forests** $F_1,F_2,\ldots,F_k$ such that
$\boldsymbol{||\partial F_i| - |\partial F_j|| \leq 2}$ $\forall i, j \in \{1,2,\ldots,k\}$



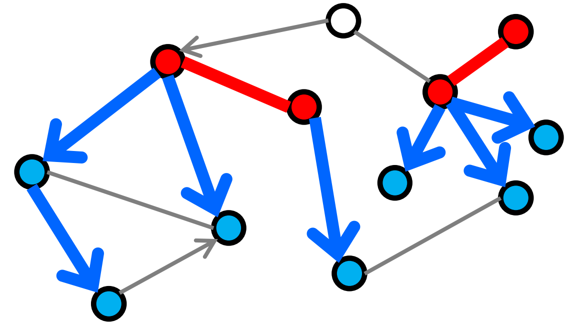
Theorem

If A can be partitioned into k $\boldsymbol{b}$**-branchings**
➔ A can be partitioned into
k $\boldsymbol{b}$**-branchings** $B_1,B_2,\ldots,B_k$ such that
- $\boldsymbol{||B_i| - |B_j|| \leq 1}$ $\forall i, j \in [k]$
- $\boldsymbol{|\mathrm{indeg}_i(v) - \mathrm{indeg}_j(v)| \leq 1}$ $\forall v\in V$, $\forall i, j \in [k]$

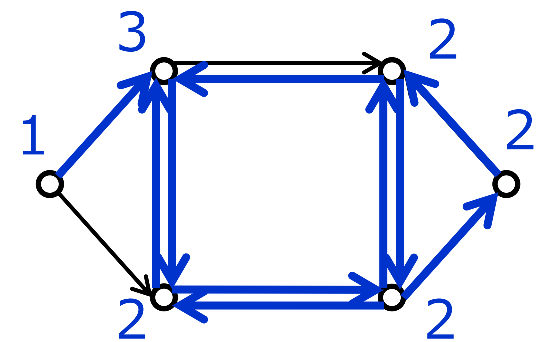


Contents

- **<u>Introduction</u>**
  - Equitable partition in graphs:
    - Matching, Branching



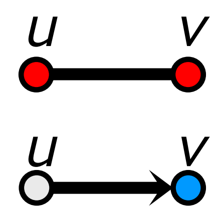
- **<u>Matching Forest</u>**
  - Common generalization of matching and branching
  - [Király, Yokoi ’18] **Tri-criteria equitability**
    - Cannot be optimized simultaneously
  - **[This talk] Single-criterion equitability**
    - Can always be optimized



- **<u>*b*-branching</u>**
  - Generalization of branchings allowing indegree ≥ 2
  - **[This talk] (n+1)-criteria equitability**
    - Can always be optimized simultaneously
- **<u>Proof Sketch</u>** (for Matching Forests)

Matching Forest [Giles ’82]

- Mixed graph $G = (V, E \cup A)$

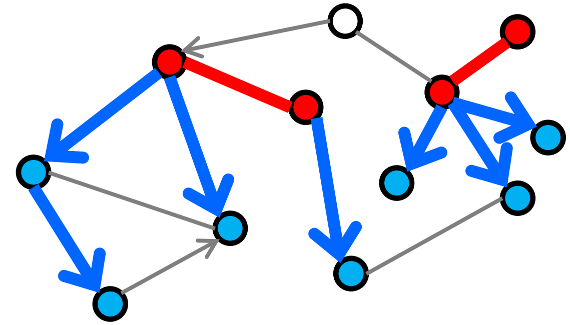


Definition

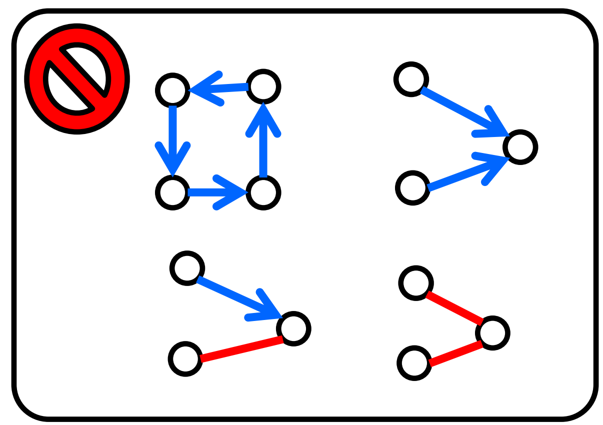
Undirected edge $\{u,v\} \in E$ **covers** both ***u*** and ***v***
Directed edge $(u,v) \in A$ **covers** only ***v***

Definition

$F \subseteq E \cup A$ is a **matching forest** if
- its underlying edge set is a forest
- every vertex is covered by ≤ 1 edge



- Namely,
  - $B = F \cap A$: Branching
  - $M = F \cap E$: Matching s.t. $\partial M \subseteq V \setminus \partial^- B$
- Set of covered vertices $\partial F := \partial^- B \cup \partial M$



Previous Work

- **Finding a max. weight matching forest** (Not a topic of today)
  - [Giles '82] Polyhedral description, Primal-dual algorithm in O(n^2m) time
  - [T. '14] Improved to O(n^3)
  - [Schrijver '00] TDI-ness of the description
  - [Schrijver '03] Reduction to weighted linear matroid parity

- **Partition into Matching Forests**
  - [Keijsper '03] Partition into Δ+1 matching forests
  - **[Király, Yokoi '18] Equitable partition into matching forests**

Theorem [Király, Yokoi '18]

If $E∪A$ can be partitioned into k matching forests

➔ $E∪A$ can be partitioned into k matching forests $F_1,F_2,\ldots,F_k$ such that

$$||F_i| - |F_j|| \leq 1, \quad ||B_i| - |B_j|| \leq 2, \quad ||M_i| - |M_j|| \leq 2$$

➔ $E∪A$ can be partitioned into k matching forests $F_1,F_2,\ldots,F_k$ such that

$$||F_i| - |F_j|| \leq 2, \quad ||B_i| - |M_j|| \leq 2, \quad ||M_i| - |M_j|| \leq 1$$

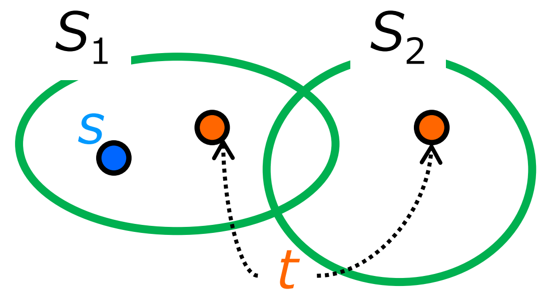
- **Tricriteria equitability**
- These values are **tight**
  i.e., the three criteria **cannot be optimized simultaneously**

Delta-Matroid Structure

- Subset family $\mathcal{F} \subseteq 2^V$

Definition

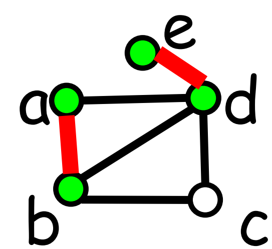
A set system $(V, \mathcal{F})$ is a **delta-matroid** if
$^\forall S_1, S_2 \in \mathcal{F},\ ^\forall s \in S_1 \Delta S_2,$
- $S_1 \Delta \{s\} \in \mathcal{F}$ or
- $^\exists t \in (S_1 \Delta S_2) - \{s\},\ S_1 \Delta \{s, t\} \in \mathcal{F}$

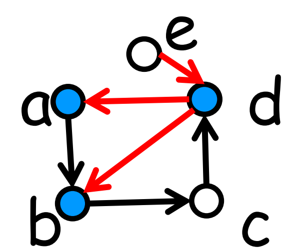


- Undirected graph $G=(V,E)$
- $\mathcal{F}_M := \{\partial M \subseteq V \mid M \text{ is a matching in } G\}$

Theorem [Chandrasekaran, Kabadi '88; Bouchet '89]

$(V, \mathcal{F}_M)$ is a **delta-matroid**





$\mathcal{F}_M$ = {Ø, {a,b}, {a,d}, {b,c}, {b,d}, {c,d}, {d,e}, {a,b,c,d}, {a,b,d,e}, {b,c,d,e}}

- Directed graph $G=(V, A)$
- $\mathcal{F}_B := \{\partial^- B \subseteq V \mid B \text{ is a branching in } G\}$

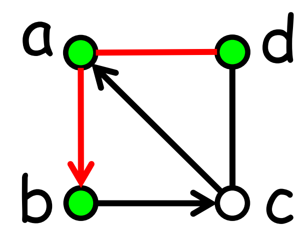
Theorem [T. '14] (Folklore?)

$(V, \mathcal{F}_B)$ is a matroid (thus a **delta-matroid**)

$\mathcal{F}_B$ = {Ø, {a}, {b}, {c}, {d}, {a,b}, {a,c},…,{c,d}, {a,b,c}, {a,b,d},{a,c,d}, {b,c,d}, {a,b,c,d}}

Equitability from the Structure

- Mixed graph $G=(V,E\cup A)$
  - $\mathcal{F}_{\mathrm{MF}} := \{\partial F \subseteq V \mid F \text{ is a matching forest in } G\}$



$\mathcal{F}_{\mathrm{MF}} = \{\emptyset, \{a\},\{b\},\{c\}, \{a,b\},\{a,c\},\{a,d\},\{b,c\}, \{c,d\},\{a,b,d\},\{a,c,d\}, \{b,c,d\}, \{a,b,c,d\}\}$

Theorem [T. '14]

$(V, \mathcal{F}_{\mathrm{MF}})$ is a **delta-matroid**

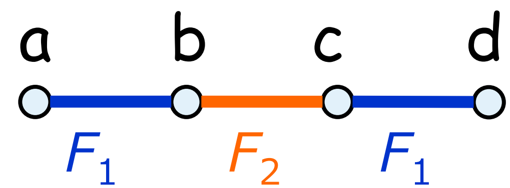
Our Idea

Define the equitability of matching forests by the size of ∂F

Theorem

If $E\cup A$ can be partitioned into k matching forests
→ $E\cup A$ can be partitioned into k matching forests $F_1,F_2,\ldots,F_k$ such that

$$||\partial F_i| - |\partial F_j|| \leq 2 \quad \forall i, j \in \{1,2,\ldots,k\}$$



$\partial F_1 = \{a,b,c,d\}$
$\partial F_2 = \{b,c\}$

- Value **2** is **optimal**
- Optimality always attained

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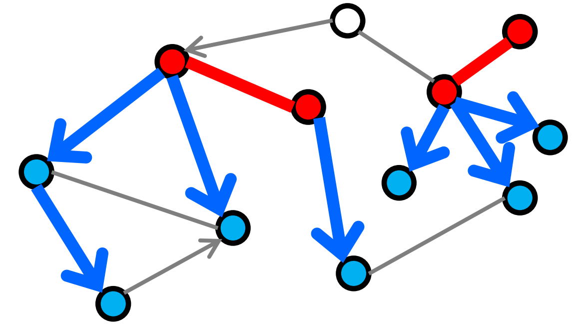

- **Matching Forest**
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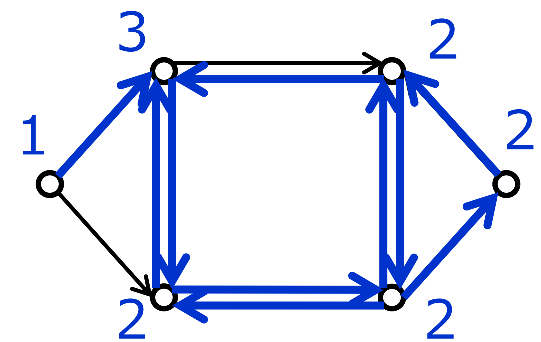
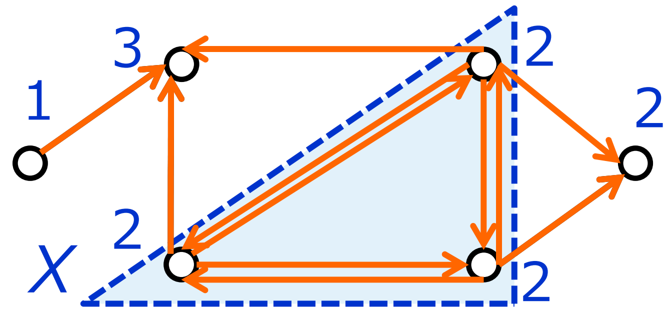


- ***b*-branching**
  - Generalization of branchings allowing indegree ≥ 2
  - **[This talk] (n+1)-criteria equitability**
    - Can always be optimized simultaneously
- **Proof Sketch** (for Matching Forests)

b-branching [Kakimura, Kamiyama, T. '20+]

- Digraph (V, A)
- Positive integer vector $\boldsymbol{b} \in \mathbf{Z}^V$ on V



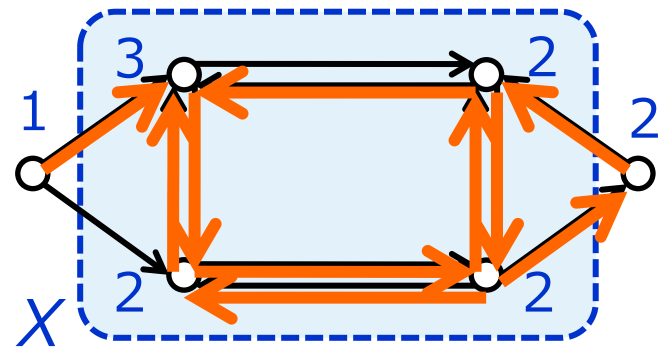
Not a b-branching:
$|B[X]|=$**6**, $b(X)$-1= **5**

Definition

$B \subseteq A$ is a ***b*-branching** if

$$\begin{cases} \text{(i) } \mathrm{indeg}_B(u) \le b(u) & (u \in V) \\ \text{(ii) } |B[X]| \le b(X) - 1 & (\emptyset \ne X \subseteq V) \end{cases}$$

- Branching: $b(u) \equiv 1$
- Special case of **matroid intersection**
  - (i): Direct sum of uniform matroids
  - (ii): Sparsity matroid (Count matroid)



A b-branching:
$|B[X]|=$**7**, $b(X)$-1=**8**

Sparsity matroid [cf. Frank's book 11]

Graph $G=(V,E)$, Vector $\boldsymbol{b} \in \mathbf{Z}^V$, Integer $k \ge 0$
➔ $\{B \subseteq E : |B[X]| \le b(X) - k\}$ is an independent set family of a matroid

Previous Work

- **Finding a max. weight *b*-branching** (Not a topic of today)
  - TDI description (← Matroid intersection polytope)
  - Multi-phase greedy algorithm extending Arborescence Algorithm [Kakimura, Kamiyama, T. '20+]

- **Partition into *b*-branchings**

Theorem [Kakimura, Kamiyama, T. '20+]

A can be partitioned into k ***b***-branchings iff

- $\mathrm{indeg}(u) \le k \cdot b(u)$ $\forall u \in V$
- $|A[X]| \le k\,(b(X) - 1)$ $\emptyset \ne \forall X \subseteq V$

- Obvious necessary condition works

Definition (Recap)

$B \subseteq A$ is a ***b*-branching** if

(i) $\mathrm{indeg}_B(u) \le b(u)$ $(u \in V)$

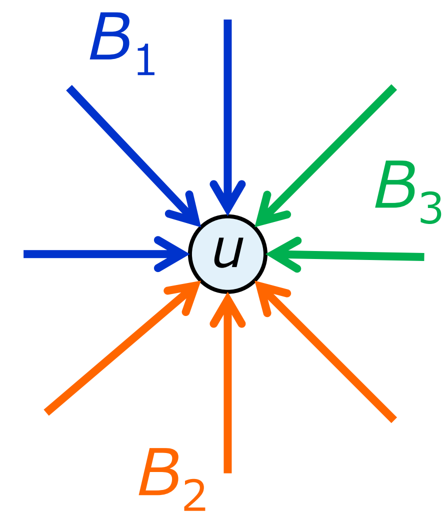
(ii) $|B[X]| \le b(X) - 1$ $(\emptyset \ne X \subseteq V)$

Equitable Partition into *b*-branchings

Theorem

If A can be partitioned into k ***b*-branchings**
➔ A can be partitioned into k ***b*-branchings** $B_1, B_2, \ldots, B_k$ such that

- $\boldsymbol{||B_i| - |B_j|| \leq 1}$ $\forall i, j \in [k]$
- $\boldsymbol{|\mathrm{indeg}_i(u) - \mathrm{indeg}_j(u)| \leq 1}$ $\forall u \in V$, $\forall i, j \in [k]$

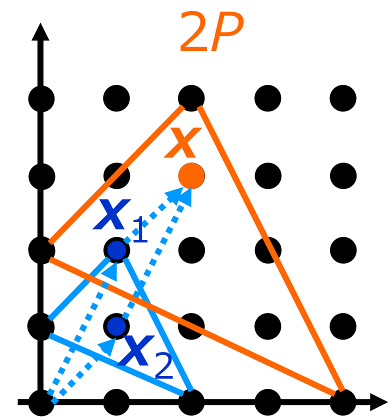


Namely, for each $i \in [k]$,

- $|B_i| = \left\lfloor \frac{|A|}{k} \right\rfloor$ or $\left\lceil \frac{|A|}{k} \right\rceil$
- $|\mathrm{indeg}_i(u)| = \left\lfloor \frac{\mathrm{indeg}_A(u)}{k} \right\rfloor$ or $\left\lceil \frac{\mathrm{indeg}_A(u)}{k} \right\rceil$ $\forall u \in V$

➔ **These $(n+1)$-criteria can be simultaneously optimized**

Integer Decomposition Property (IDP)



Definition

A polytope P has the **integer decomposition property** if $\forall k \in \mathbf{Z}_{++}$, $\forall \boldsymbol{x} \in kP \cap \mathbf{Z}^A$, $\boldsymbol{x} = \boldsymbol{x}_1 + \cdots + \boldsymbol{x}_k$ ($\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in P \cap \mathbf{Z}^A$)

- **True** for
  - Polymatroids [Giles '75; Baum, Trotter '81]
  - Intersection of two strongly base orderable matroids [Davies, McDiarmid '76; McDiarmid '76]
  - Branching polytope (below)
- **NOT True** for every matroid intersection

Theorem [Baum, Trotter '81]

The **branching polytope** has IDP

Theorem [McDiarmid '83]

The convex hull of the incidence vectors of the **branchings of size ℓ** has IDP

Branching polytope

- $x(\delta u) \leq 1$ $\quad (u \in V)$
- $x(A[X]) \leq |X| - 1$ $\quad (\emptyset \neq X \subseteq V)$
- $0 \leq x(a) \leq 1$ $\quad (a \in A)$

IDP for b-branchings

[Baum, Trotter '81]

Branching

[Kakimura, Kamiyama, T. '20+]

The ***b*-branching polytope** has **IDP**

[McDiarmid '83]

Branchings of size ℓ

[Our Result]

The convex hull of the incidence vectors of the ***b*-branchings of size ℓ** has **IDP**

- Further, the indegree can be fixed to be $b'(v)$ ($\leq b(v)$)

[Our Result]

The convex hull of the incidence vectors of the ***b***-branchings of

- **size ℓ**; and
- **indeg(v)=$b'(v)$ $\forall v \in V$**

has **IDP**

***b*-branching polytope**

- $x(\delta u) \leq b(u)$ $(u \in V)$
- $x(A[X]) \leq b(X) - 1$ $(\emptyset \neq X \subseteq V)$
- $0 \leq x(a) \leq 1$ $(a \in A)$

Contents

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  - Equitable partition in graphs:
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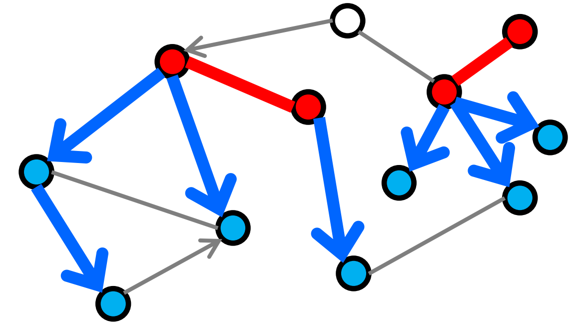

- **Matching Forest**
  - Common generalization of matching and branching
  - [Király, Yokoi '18] **Tri-criteria equitability**
    - Cannot be optimized simultaneously
  - **[This talk] Single-criterion equitability**
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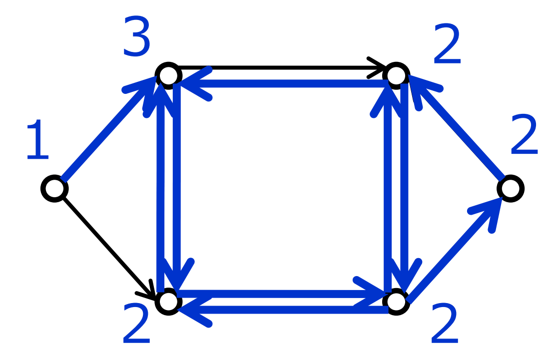


- ***b*-branching**
  - Generalization of branchings allowing indegree ≥ 2
  - **[This talk] (n+1)-criteria equitability**
    - Can always be optimized simultaneously

- **Proof Sketch** (for Matching Forests)

Key Lemma

- Branching B
- Root set $R(B) = V \setminus \partial^{-} B$

Key Lemma [Schrijver '00]

Branchings B_1, B_2 partitioning A

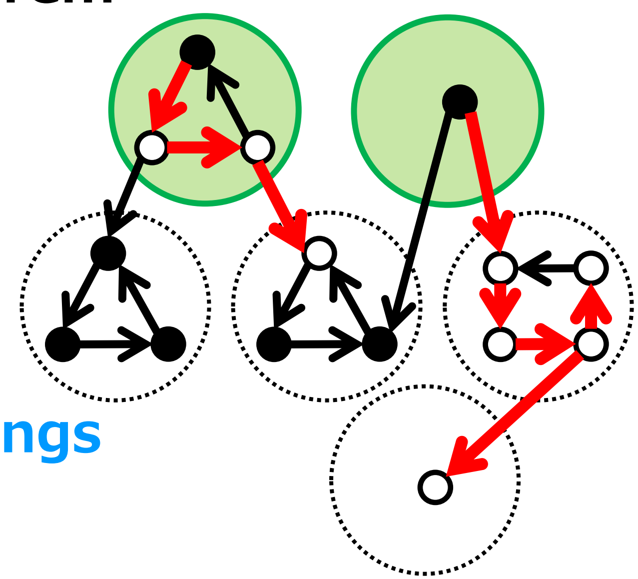
$X, Y \subseteq V$ satisfying $X \cup Y = R(B_1) \cup R(B_2)$, $X \cap Y = R(B_1) \cap R(B_2)$

➔ $^{\exists}$Branchings B_1', B_2' partitioning A such that $R(B_1') = X$, $R(B_2') = Y$ iff

$|X \cap C| \geq 1$ and $|Y \cap C| \geq 1$

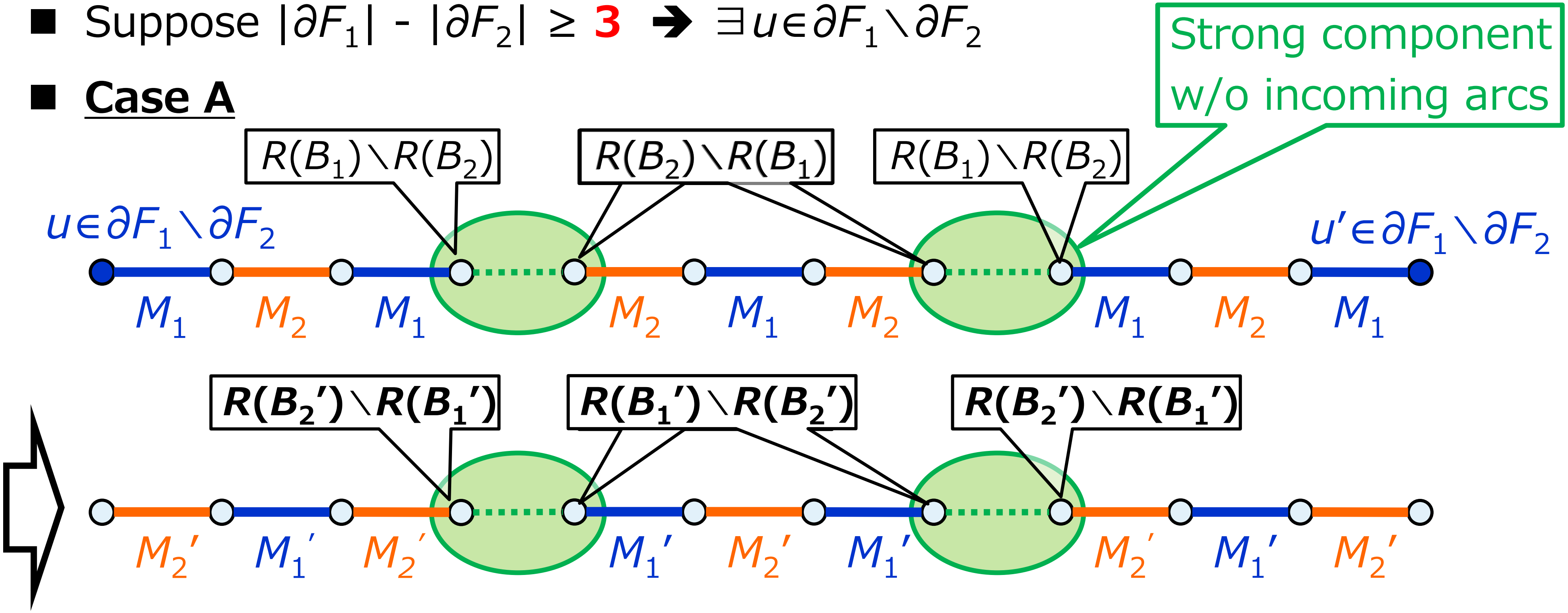
for each strong component C w/o incoming arcs

- Variant of **Edmonds' disjoint branchings theorem**
- All derives
  - TDI-ness of matching forests
  - Delta-matroid structure of matching forests
  - **Equitable partition into matching forests**
- Extended to $\boldsymbol{b}$-branchings
  - Derives **equitable partition into $\boldsymbol{b}$-branchings**



Proof Sketch for Matching Forests (1)

- Suppose $|\partial F_1|$ - $|\partial F_2| \geq$ **3** ➔ $\exists u \in \partial F_1 \setminus \partial F_2$
- **Case A**

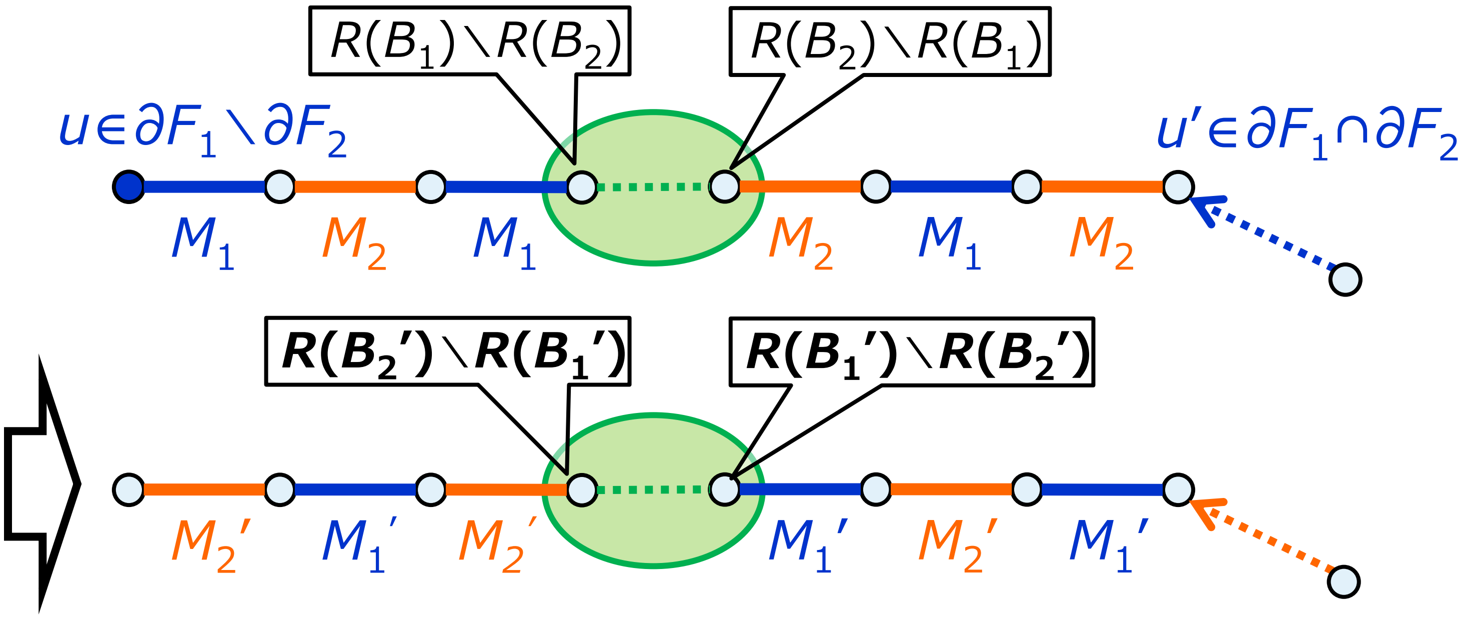


➢ Key Lemma shows the existence of B_1', B_2'

➢ $|\partial F_1|$ - $|\partial F_2|$ decreases by **4**

Proof Sketch for Matching Forests (2)

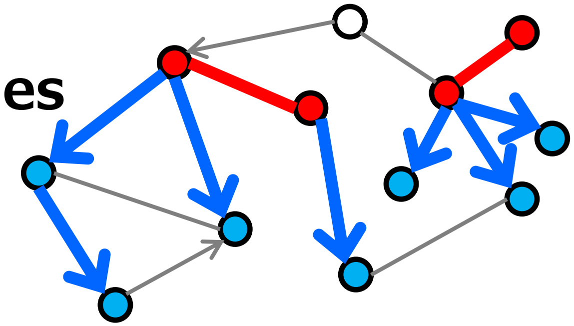
- Suppose $|\partial F_1|$ - $|\partial F_2| \geq$ **3** ➔ $\exists u \in \partial F_1 \setminus \partial F_2$
- **Case B**



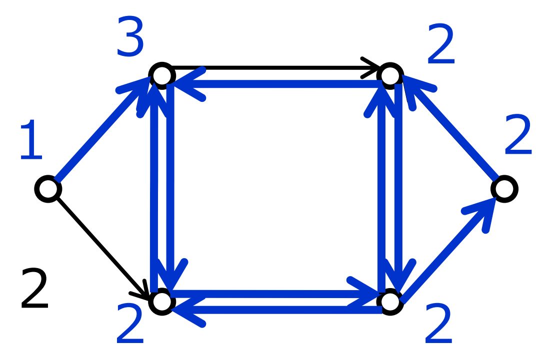
- Key Lemma shows the existence of B_1', B_2'
- $|\partial F_1|$ - $|\partial F_2|$ decreases by **2**

Conclusion

- **Matching Forest**
  - Common generalization of matching and branching
  - [Király, Yokoi ’18] **Tricriteria equitability**
    - **Number of directed/undirected/all edges**
    - Cannot be optimized simultaneously
  - [This talk] **Single-criterion equitability**
    - **Number of covered vertices**
    - Can always be optimized



- ***b*-branching**
  - Generalization of branchings allowing indegree ≥ 2
  - [This talk] **(*n*+1)-criteria equitability**
    - **Number of all edges + indegree of each vertex *v***
    - Can always be optimized simultaneously



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