The *b*-branching problem in digraphs

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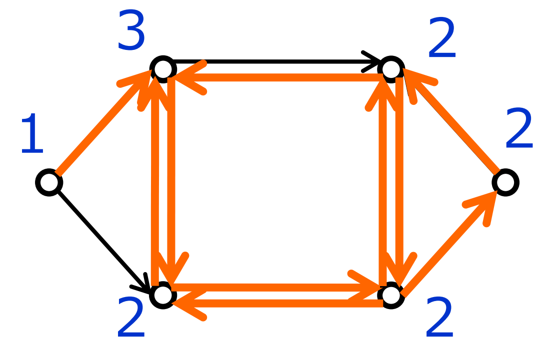

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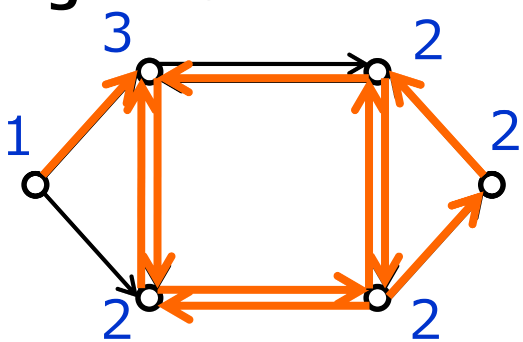
Overview

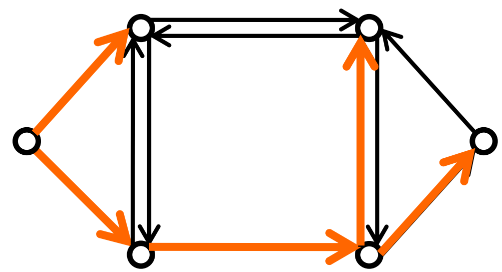
b-branching [This talk]

- **Algorithm**
- **Packing theorem**



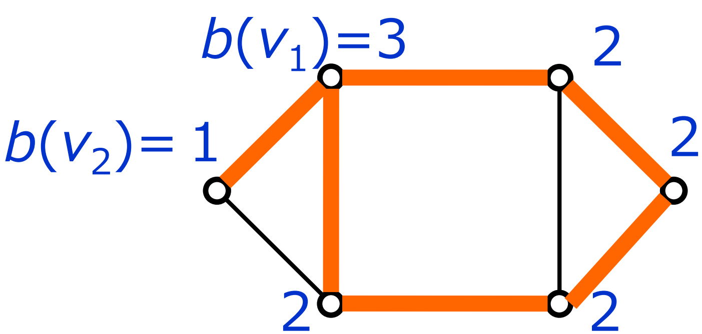
Branching

- Algorithm [Chu & Liu 65, etc.]
- Packing theorem [Edmonds 73]



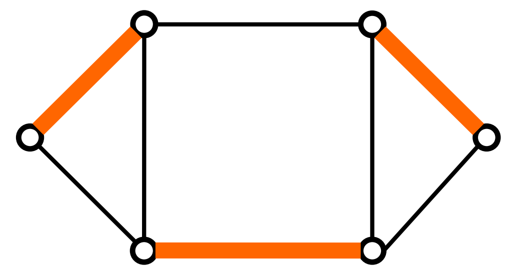
b-matching

- Min-max theorem [Tutte 54]
- Algorithm [Marsh 79]



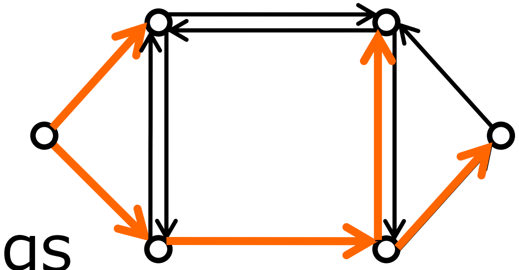
Matching

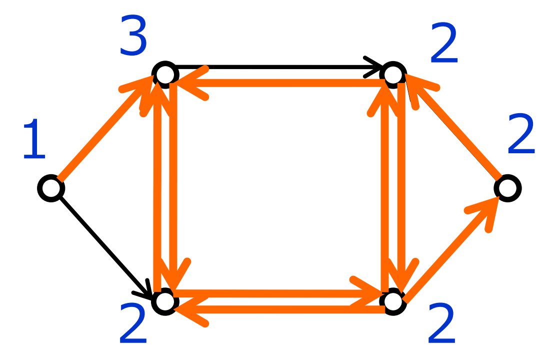
- Min-max theorem [Tutte 47]
  [Berge 58]
- Algorithm [Edmonds 65]



Contents

- <u>Introduction: Branching</u>
  - ■ Partition matroid ∩ Graphic matroid
  - ➢ Algorithm
  - ➢ Theorem for packing disjoint branchings

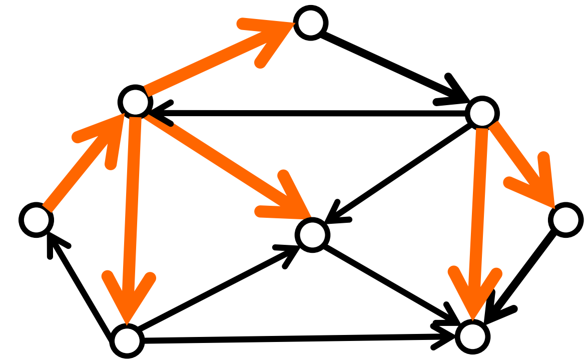




- **<u>Our result: *b*-branching</u>**
  - ■ Partition matroid ∩ Sparsity matroid
  - ➢ **Algorithm**
  - ➢ **Theorem for packing disjoint *b*-branchings**

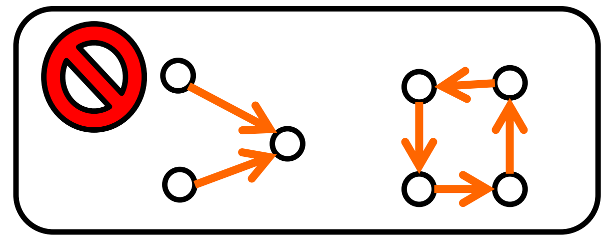
Branching

◆ Digraph (V, A)



Definition

- $B \subseteq A$ is a **branching** $\Leftrightarrow$
  - (i) $\mathrm{indeg}(u) \leq 1 \quad (u \in V)$
  - (ii) No undirected cycle



Fact

Special case of **Matroid intersection**

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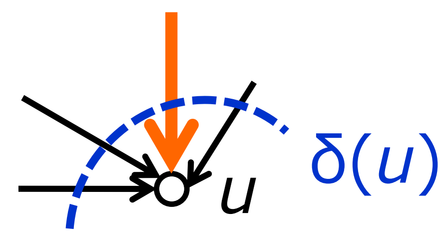
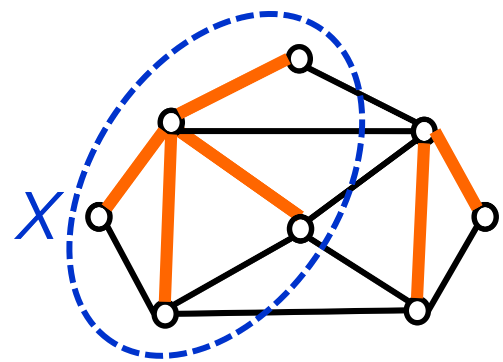


- (i): **Partition matroid**
  - ✓ $A = \delta^{\mathrm{in}}(u_1) \cup \delta^{\mathrm{in}}(u_2) \cup \cdots \cup \delta^{\mathrm{in}}(u_n)$
  - ✓ $|B \cap \delta^{\mathrm{in}}(u_i)| \leq 1 \quad (i = 1, 2, \ldots, n)$
- (ii): **Graphic matroid**
  - ✓ $|B[X]| \leq |X| - 1 \ (\emptyset \neq X \subseteq V)$



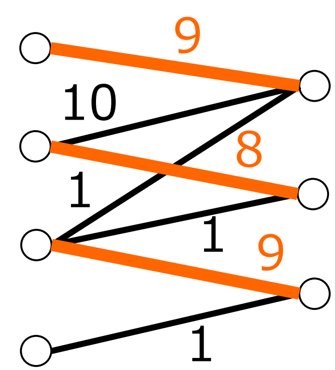
Matroid intersection

- Matroids $\mathbf{M}_1=(A, \mathcal{I}_1)$, $\mathbf{M}_2=(A, \mathcal{I}_2)$
- Weight $w\in\mathbf{R}^A$

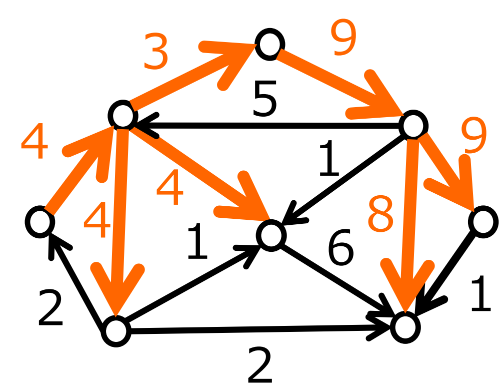
Weighted matroid intersection problem

- Find $B \in\mathcal{I}_1\cap\mathcal{I}_2$ maximizing $w(B)=\sum_{a\in B} w(a)$

- **Solved in polynomial time**
  - ✓ Augmenting-path algorithm [Edmonds 70]



- Example 1 : **Bipartite matching**
  - ✓ $\mathbf{M}_1$: Partition matroid
  - ✓ $\mathbf{M}_2$: Partition matroid



- Example 2 : **Branching**
  - ✓ $\mathbf{M}_1$: Partition matroid
  - ✓ $\mathbf{M}_2$: Graphic matroid

Branching as Special Matroid Intersection

◆ Properties (A),(B): Not true for general matroid intersection

Property (A)

Greedy-type algorithm for max weight branching
[Chu-Liu 65, Edmonds 67, Bock 71, Fulkerson 74]

- **NOT** true for **bipartite matching (!)**

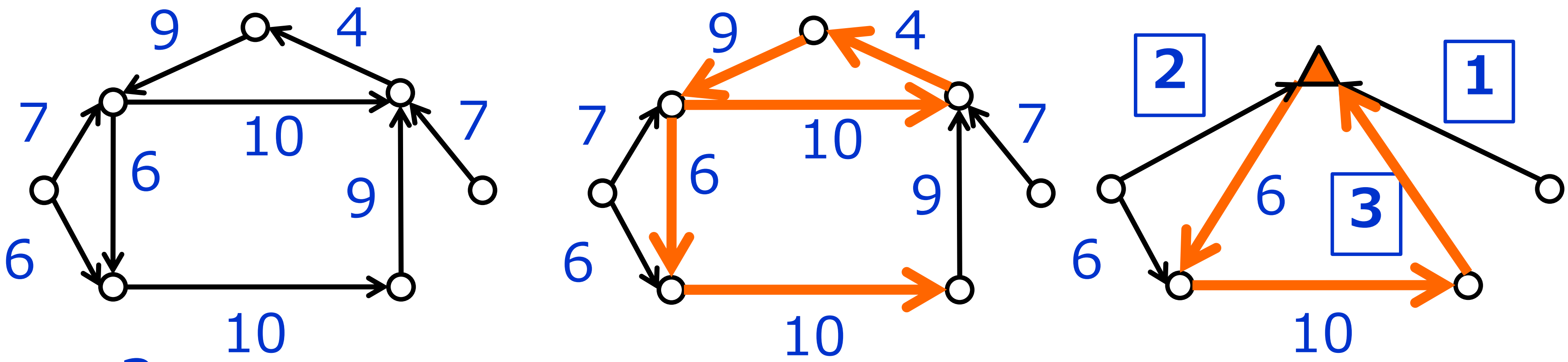
Property (B)

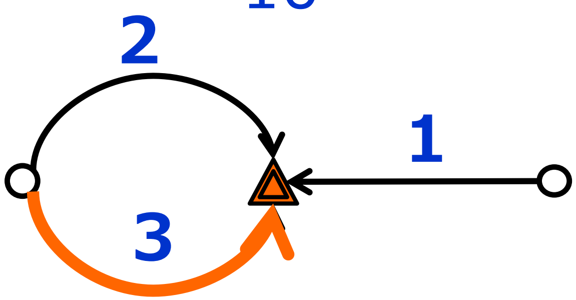
Theorem for packing disjoint branchings [Edmonds 73]

- Also holds for
  - Bipartite matching (Kőnig's edge-coloring theorem)
  - Strongly base orderable matroid intersection [Davies, McDiarmid 76]
  - Matroids without (k + 1)-spanned elements [Kotlar, Ziv 05]
    [**T.**, Yokoi 18]

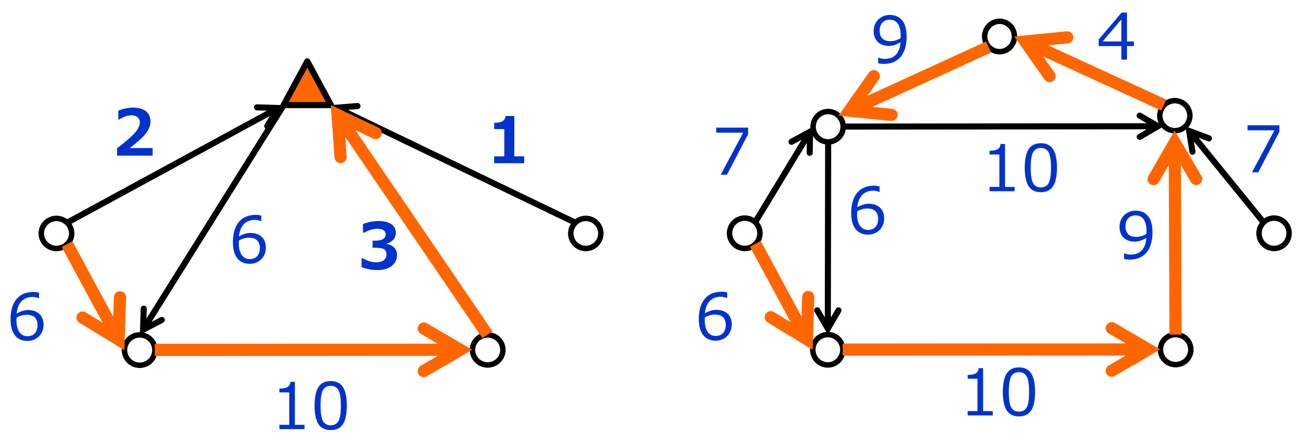
(A) Greedy-type Algorithm

[Chu-Liu 65, Edmonds 67, Bock 71, Fulkerson 74]





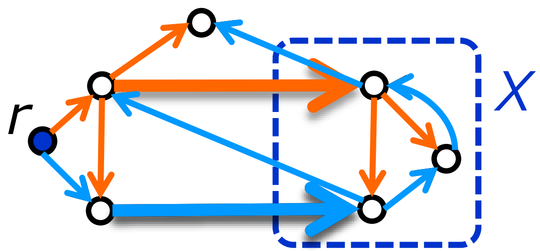
2 = 7 + 4 − 9
3 = 9 + 4 − 10
1 = 7 + 4 − 10



(B) Packing Disjoint Branchings

Theorem [Edmonds 67, Bock 71, Fulkerson 74]

Digraph D has **one** r-arborescence
$\Leftrightarrow |\delta^{in}(X)| \geq 1 \quad (\emptyset \neq X \subseteq V \setminus \{r\})$



Disjoint Arborescences Theorem [Edmonds 73]

Digraph D has ***k* arc-disjoint** r-arborescence
$\Leftrightarrow |\delta^{in}(X)| \geq k \quad (\emptyset \neq X \subseteq V \setminus \{r\})$

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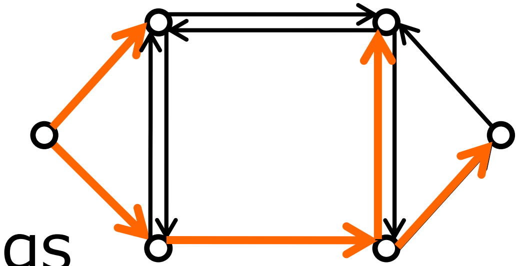
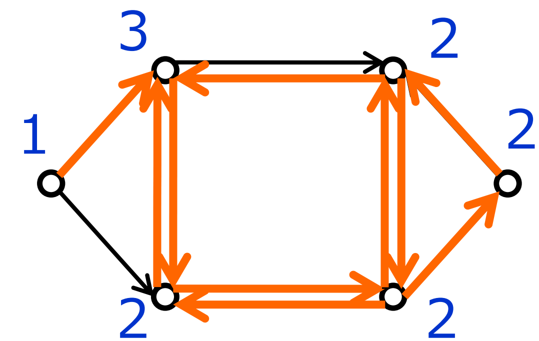
    - Algorithm
    - Theorem for packing disjoint branchings



- **<u>Our result: *b*-branching</u>**
  - Partition matroid ∩ Sparsity matroid
    - **Algorithm**
    - **Theorem for packing disjoint *b*-branchings**



b-branching

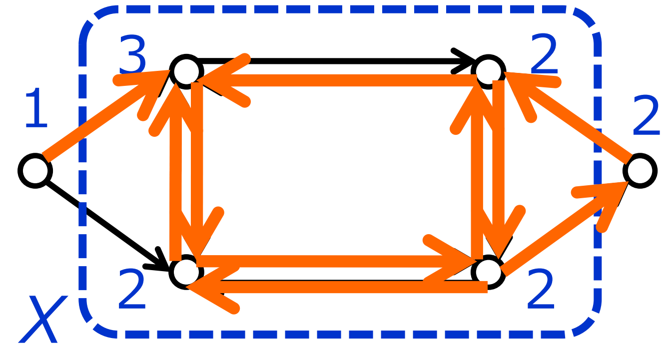
- Digraph (V, A)
- Positive integer vector $\boldsymbol{b} \in \mathbf{Z}^V$ on V

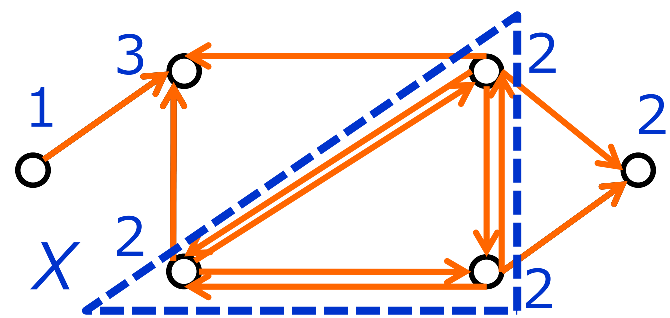
Definition

- $B \subseteq A$ is a ***b*-branching** $\Leftrightarrow$

$$\begin{cases} \text{(i) } \mathrm{indeg}_B(u) \le b(u) & (u \in V) \\ \text{(ii) } |B[X]| \le b(X) - 1 & (\emptyset \ne X \subseteq V) \end{cases}$$

- Branching: $b(u)=1$





- (i): Direct sum of **uniform matroids**
- (ii): **Sparsity matroid**

Sparsity matroid [cf. Frank 11]

Graph $G=(V,E)$, Vector $\boldsymbol{b} \in \mathbf{Z}^V$, Integer $k \ge 0$

- $\{B \subseteq E : |B[X]| \le b(X) - k\}$ is an independent set family of a matroid

✓ k disjoint branchings: $\mathrm{indeg}(u) \le k \quad (u \in V)$

$|B[X]| \le k|X| - k \quad (\emptyset \ne U \subseteq V)$

Our Results: Properties of *b*-branching

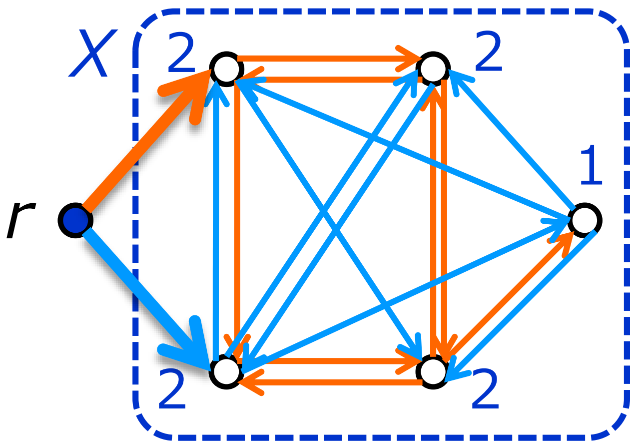
Property (A) [Our result]

Greedy-type algorithm for max weight *b*-branching

- More tractable than **bipartite matching**

Property (B) [Our result]

Theorem for packing disjoint *b*-branchings
\+ **Integer decomposition property** of the polytope



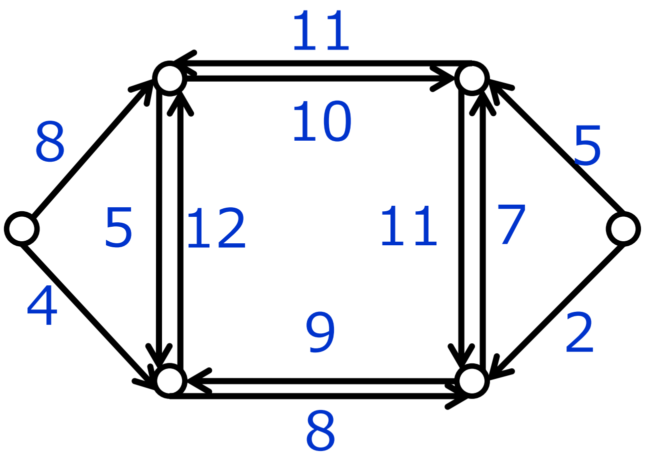
***b*-branching polytope**

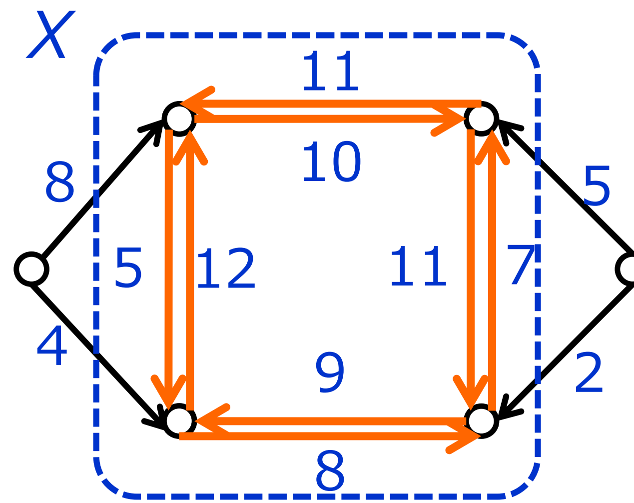
$$x(\delta(u)) \le b(u) \quad (u \in V)$$
$$x(A[X]) \le b(X) - 1 \quad (\emptyset \ne X \subseteq V)$$
$$x(a) \ge 0 \quad (a \in A)$$

- $|\delta(u)| \ge k \cdot b(u) \quad (u \in V \setminus \{r\})$
- $|\delta(X)| \ge k \quad (\emptyset \ne X \subseteq V \setminus \{r\})$

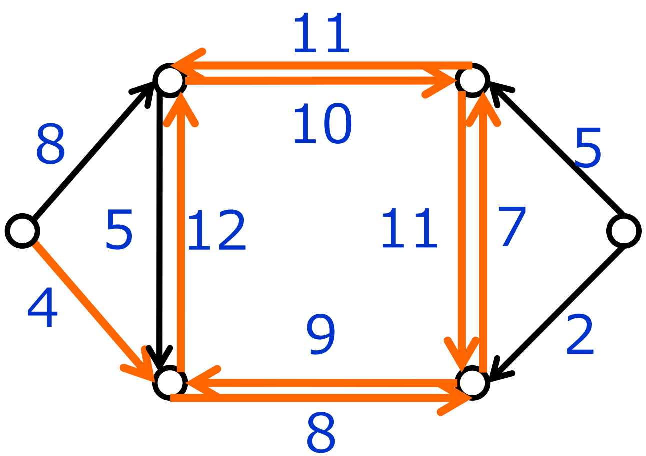
(A) Greedy-type Algorithm

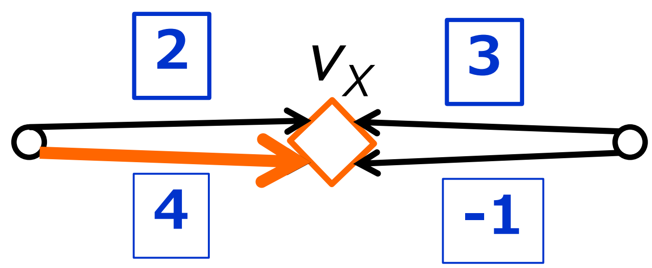
$b(u)=2\ (^\forall u \in V)$





$|B[X]|=b(X)$





$b(v_X) := 1$

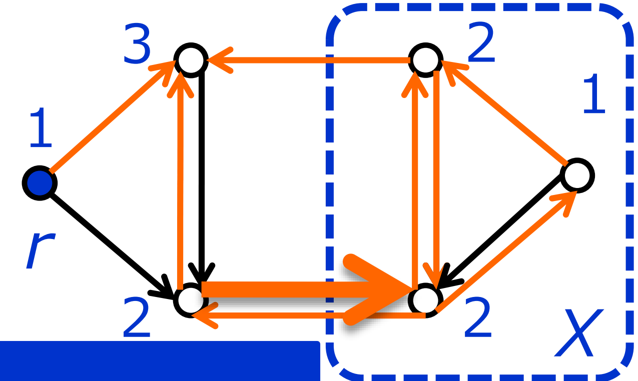
2 = 8 + 5 − 11
4 = 4 + 5 − 5
3 = 5 + 5 − 7
-1 = 2 + 5 − 8

(B) Packing Disjoint *b*-branchings

Theorem

b-branching with indeg(u)=$b(u)$ ($u \in V \setminus \{r\}$) ⇔

- $|\delta^{in}(u)| \geq b(u)$ ($u \in V \setminus \{r\}$)
- $|\delta^{in}(X)| \geq 1$ ($\emptyset \neq X \subseteq V \setminus \{r\}$)



Theorem

k* disjoint *b-branchings with indeg(u)=$b(u)$ ($u \in V \setminus \{r\}$) ⇔

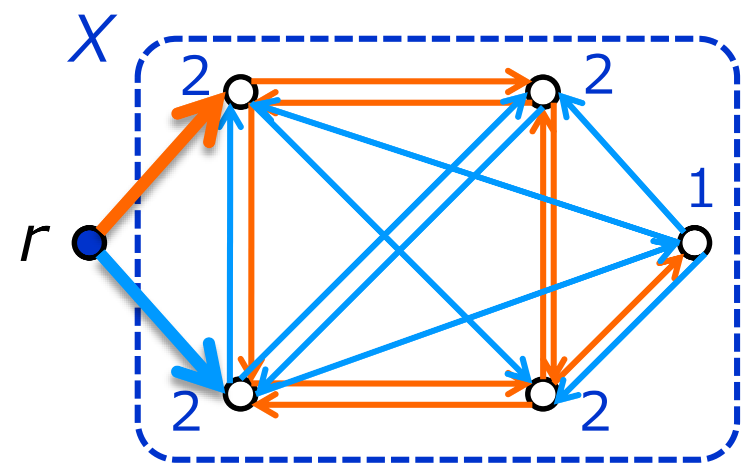
- $|\delta^{in}(u)| \geq k \cdot b(u)$ ($u \in V \setminus \{r\}$)
- $|\delta^{in}(X)| \geq k$ ($\emptyset \neq X \subseteq V \setminus \{r\}$)

Th. [Edmonds 67, Bock 71, Fulkerson 74]

r-arborescence
⇔ $|\delta^{in}(X)| \geq 1$ ($\emptyset \neq X \subseteq V \setminus \{r\}$)

Th. [Edmonds 73]

***k* disjoint** *r*-arborescence
⇔ $|\delta^{in}(X)| \geq k$ ($\emptyset \neq X \subseteq V \setminus \{r\}$)



Algorithm for Packing *b*-branchings

Theorem [Recap]

k disjoint ***b***-branchings with indeg(u)=$b(u)$ ($u \in V \setminus \{r\}$) exist ⇔

- $|\delta(u)| \geq k \cdot b(u)$ $(u \in V \setminus \{r\})$
- $|\delta(X)| \geq k$ $(\emptyset \neq X \subseteq V \setminus \{r\})$

Theorem

Packing ***b***-branchings in **strongly polynomial time**:

① Existence
② Find a packing
③ Find a minimum-cost packing

① **Min-cut** algorithm
② **Min-cut** algorithm $k \cdot b(V) \cdot |A|$ ($\leq |A|^2$) times
③ **Submodular-flow** algorithm

Integer Decomposition Property

Theorem [Recap]

k disjoint $\boldsymbol{b}$-branchings with indeg$(u)=b(u)$ $(u\in V\setminus\{r\})$ $\Leftrightarrow$

- $|\delta(u)| \geq k\cdot b(u)$ $\quad (u \in V\setminus\{r\})$
- $|\delta(X)| \geq k$ $\quad (\emptyset \neq X \subseteq V\setminus\{r\})$

Theorem

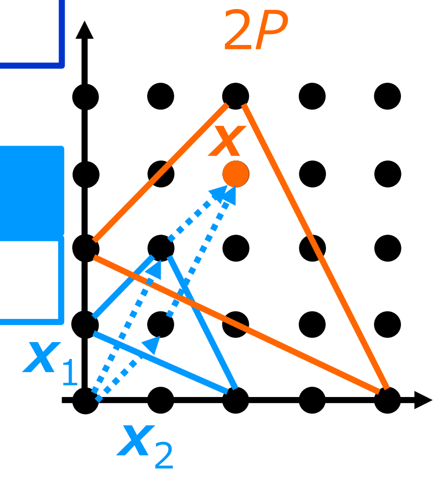
The $\boldsymbol{b}$-branching polytope:

- $x(\delta u) \leq b(u)$ $\quad (v\in V)$
- $x(A[X]) \leq b(X) - 1$ $\quad (\emptyset\neq X\subseteq V)$
- $0 \leq x(a) \leq 1$ $\quad (a\in A)$

has the **integer decomposition property**

Integer decomposition property of a polytope *P*

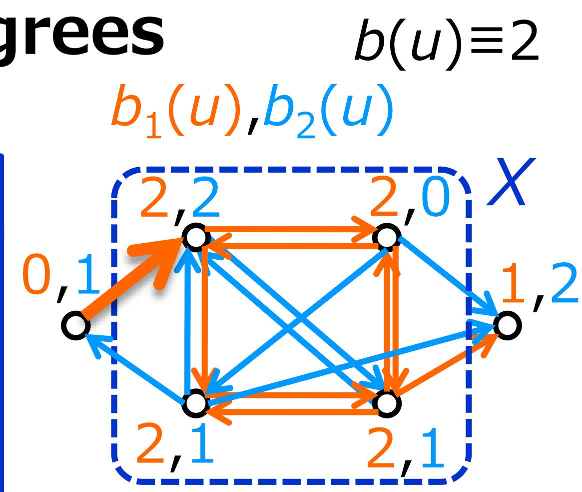
$^\forall k\in\mathbf{Z}_{++}$, $^\forall \boldsymbol{x}\in kP\cap\mathbf{Z}^A$, $\boldsymbol{x} = \boldsymbol{x}_1+\cdots+\boldsymbol{x}_k$ $(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k\in P\cap\mathbf{Z}^A)$



Further Generalizations

- Packing ***b***-branchings with **different indegrees**

$\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_k \in \mathbf{Z}^V$ (with $\boldsymbol{b}_i \leq \boldsymbol{b}$)



Theorem

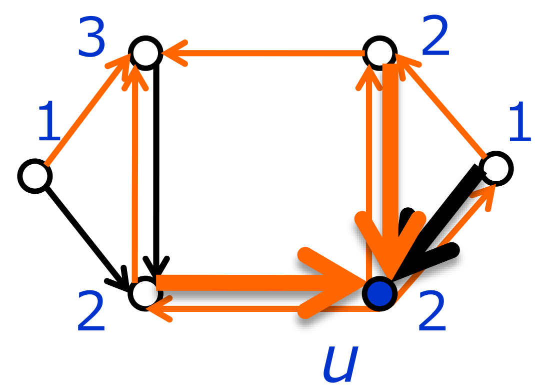
Disjoint ***b***-branchings $B_1, B_2, \ldots, B_k$
with $\mathrm{indeg}_{B_i}(u) = b_i(u)$ $(u \in V,\ i = 1, 2, \ldots, k)$ ⇔

- $|\delta(v)| \geq \sum_{i=1}^{k} b_i(v)$ $\quad (v \in V)$
- $|\delta(X)| \geq |\{i \in [k] : b_i(X) = b(X) \neq 0\}|$ $\quad (\emptyset \neq X \subseteq V)$

➔ $|\delta(X)| \geq 1$

- **Matroid-restricted *b*-branching**

Each $u \in V$ associated with a matroid $\mathbf{M}_u = (\delta(u), \mathcal{I}_u)$



Definition

- $B \subseteq A$ is a **matroid-restricted *b*-branching** ⇔

$$\begin{cases} \text{(i) } \boldsymbol{B \cap \delta(u)} \in \mathcal{I}_u & (u \in V) \\ \text{(ii) } |B[X]| \leq b(X) - 1 & (\emptyset \neq X \subseteq V) \end{cases}$$

- Greedy-type algorithm can be extended

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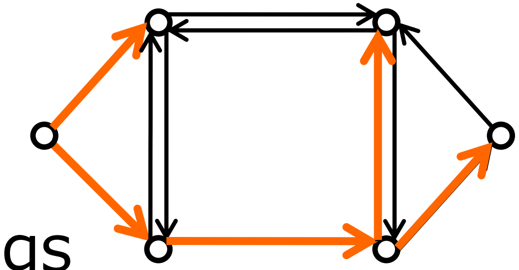
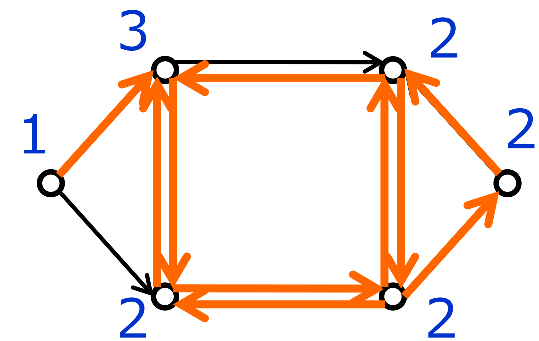


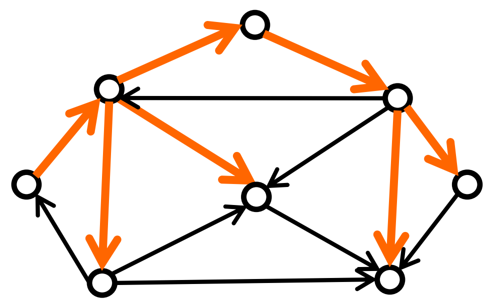
- **Our result: *b*-branching**
  - ■ Partition matroid ∩ Sparsity matroid
  - ➢ **Algorithm**
  - ➢ **Theorem for packing disjoint *b*-branchings**

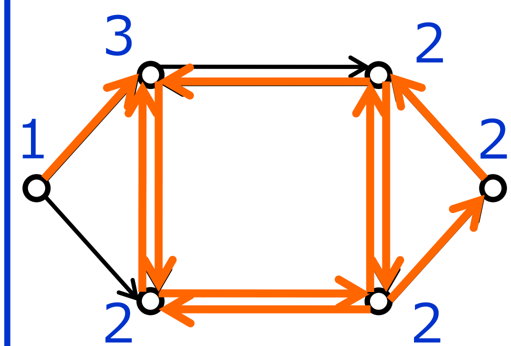


Summary & Future Work

Summary

- ***b*-branchings** in digraphs
  - **indeg(*u*) ≤ *b*(*u*)**
    - ✓ Matching ➔ ***b***-matching
    - ✓ Branching ➔ ***b***-branching
  - Other matroid = **Sparsity matroid**
  - Heritage from branchings:
    - ✓ **Greedy-type algorithm**
    - ✓ **Packing theorem**
    - ✓ **Integer decomposition property**





Future work

- Other class of matroid intersection admitting greedy-type algorithm ?
- Extending other results on branchings ?

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