# The *b*-bibranching Problem:
## TDI system, Packing, and Discrete Convexity

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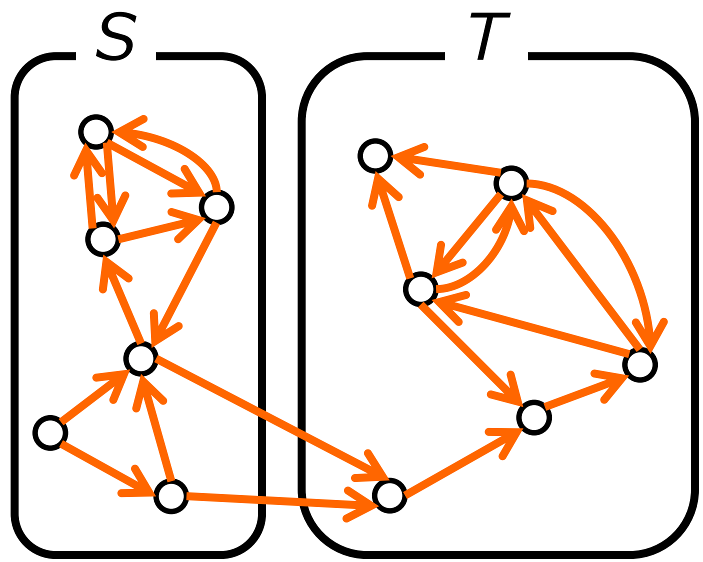


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# Outline



(3) ***b***-bibranching

**Our result**

- TDI system
- Packing
- M-convex submodular flow formulation

(2-2) Bibranching

- TDI system [Schrijver 82]
- Packing [Schrijver 82]
- M-convex submodular flow formulation [**T.** 12]

(2-1) ***b***-branching

Counterpart of ***b***-matching

- TDI system [Kakimura, Kamiyama, **T.** 18]
- Packing [Kakimura, Kamiyama, **T.** 18]

(1) Branching

- TDI system [Edmonds 70]
- Packing [Edmonds 73]

# Branching

◆ Digraph $(V, A)$

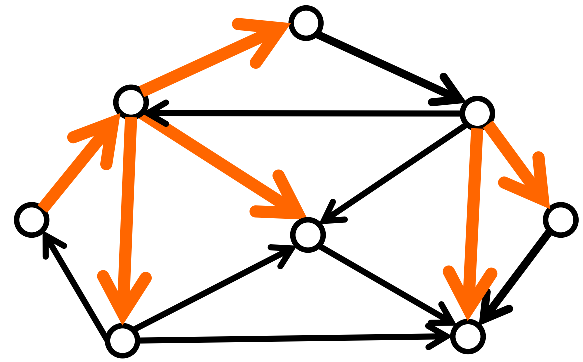
**Definition**

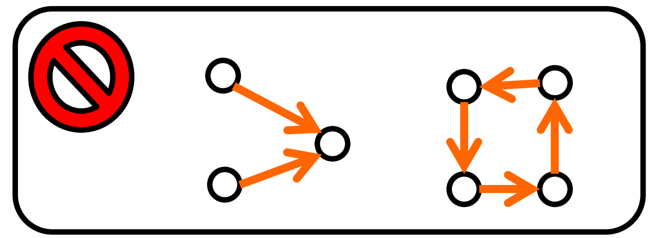
- $B \subseteq A$ is a **branching** $\Leftrightarrow$
  - (i) $\mathrm{indeg}_B(u) \leq 1 \quad (u \in V)$
  - (ii) No undirected cycle

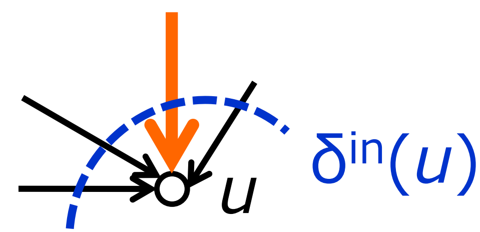
**Fact**

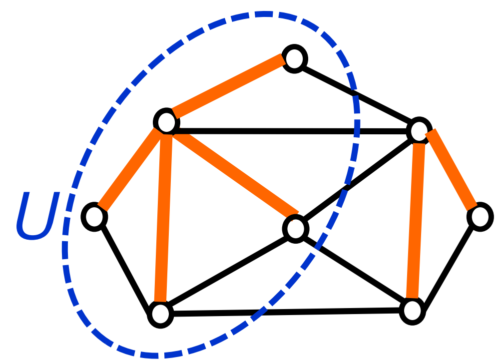
Branching: A special case of **matroid intersection**

- (i): **Partition matroid**
  - ✓ $A = \delta^{\mathrm{in}}(u_1) \cup \delta^{\mathrm{in}}(u_2) \cup \cdots \cup \delta^{\mathrm{in}}(u_n)$
  - ✓ $|B \cap \delta^{\mathrm{in}}(u_i)| \leq 1 \quad (i=1,2,\ldots,n)$
- (ii): **Graphic matroid**
  - ✓ $|B[U]| \leq |U| - 1 \quad (\emptyset \neq U \subseteq V)$







# Important Results on Branchings



**Result (A)**

**TDI linear system**

- Follows from TDIness of matroid intersection [Edmonds 70]

**Result (B)**

**Multi-phase greedy algorithm** for max weight branching [Chu-Liu 65, Edmonds 67, Bock 71, Fulkerson 74]

- **NOT** true for **bipartite matching (!)**

**Result (C)**

**Packing theorem** [Edmonds 73]

- Also holds for
  - Bipartite matching (Kőnig's theorem)
  - Strongly base orderable matroid intersection [Davies, McDiarmid 76]
  - Matroids without ($k$ + 1)-spanned elements [Kotlar, Ziv 05] [**T.**, Yokoi 18]

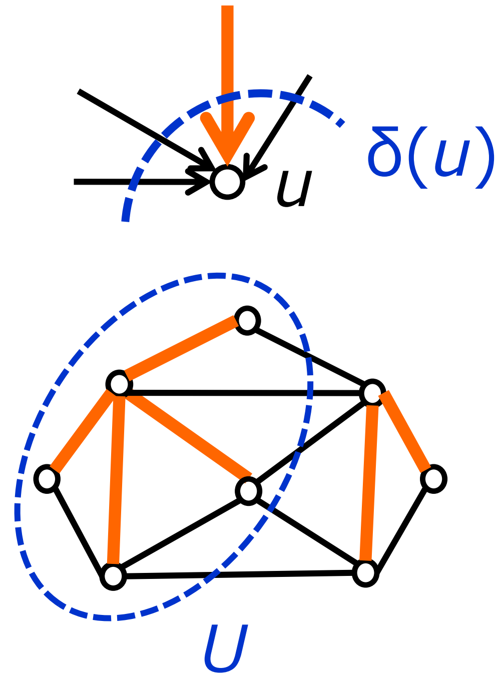
# (A) TDI System for Branchings



**Linear Program (P)** in variable $x \in \mathbf{R}^A$

max. $\sum w(a)x(a)$

s.t. $x(\delta^{in}(u)) \leq 1 \quad (u \in V)$
$x(A[U]) \leq |U| - 1 \quad (\emptyset \neq U \subseteq V)$
$x(a) \geq 0 \quad (a \in A)$




**TDI Theorem** [Edmonds 70]

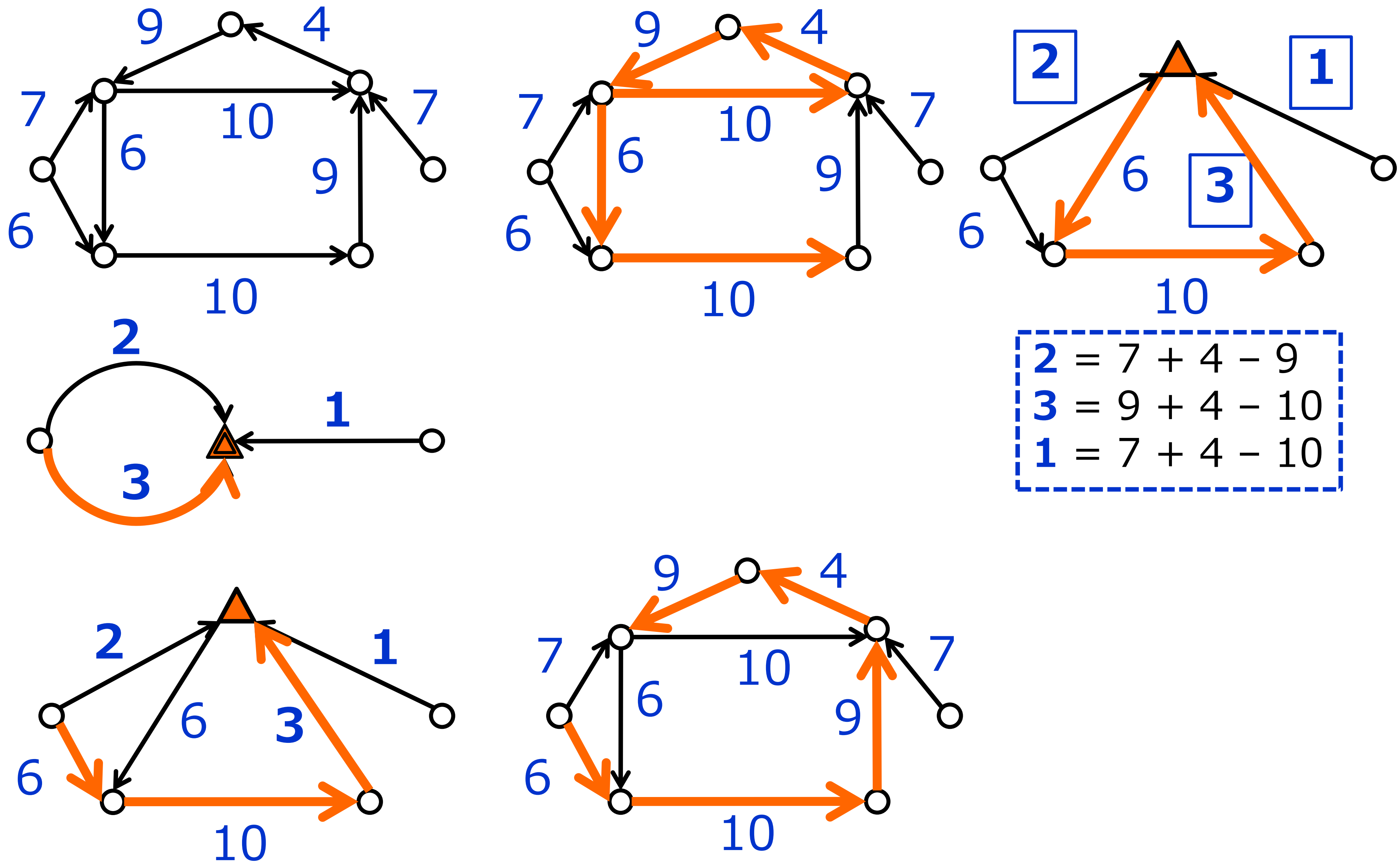
**This linear system is TDI**, i.e.,

- (P) has an integer optimal solution
- If $w$ is integer, the dual program also has an integer optimal solution

- Holds for any matroid intersection [Edmonds 70]

# (B) Multi-phase Greedy Algorithm

[Chu-Liu 65, Edmonds 67, Bock 71, Fulkerson 74]

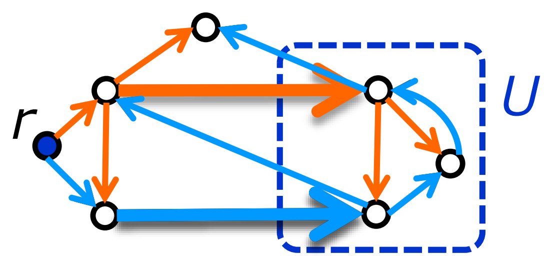



$$\mathbf{2} = 7 + 4 - 9$$
$$\mathbf{3} = 9 + 4 - 10$$
$$\mathbf{1} = 7 + 4 - 10$$

# (C) Packing Disjoint Branchings

**Theorem** [Edmonds 67, Bock 71, Fulkerson 74]

Digraph $D$ has **one** $r$-arborescence
$\Leftrightarrow$ $|\delta^{in}(U)| \geq 1$ $(\emptyset \neq U \subseteq V \setminus \{r\})$




**Disjoint Arborescences Theorem** [Edmonds 73]

Digraph $D$ has ***k* arc-disjoint** $r$-arborescence
$\Leftrightarrow$ $|\delta^{in}(U)| \geq k$ $(\emptyset \neq U \subseteq V \setminus \{r\})$

# Outline



(3) ***b***-bibranching

**Our result**

- TDI system
- Packing
- M-convex submodular flow formulation

(2-2) Bibranching

- TDI system [Schrijver 82]
- Packing [Schrijver 82]
- M-convex submodular flow formulation [**T.** 12]

(2-1) ***b***-branching

- TDI system [Kakimura, Kamiyama, **T.** 18]
- Packing [Kakimura, Kamiyama, **T.** 18]

(1) Branching

- TDI system [Edmonds 70]
- Packing [Edmonds 73]

# *b*-branching

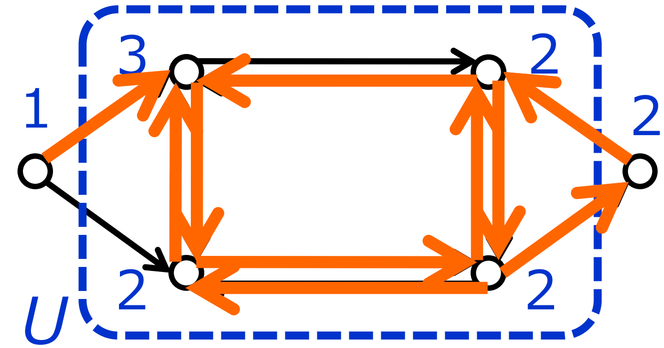
- ◆ Digraph $(V, A)$
- ◆ Positive integer vector $\boldsymbol{b} \in \mathbf{Z}^V$ on $V$

**Def.** **[Kakimura, Kamiyama, T. 18]**

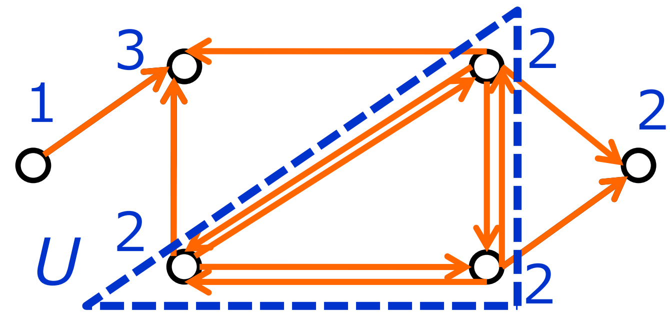
- ● $B \subseteq A$ is a ***b*-branching** ⇔

$$\begin{cases} \text{(i) } \mathrm{indeg}_B(u) \le b(u) & (u \in V) \\ \text{(ii) } |B[U]| \le b(U) - 1 & (\emptyset \ne U \subseteq V) \end{cases}$$

- • Branching: $b(u)=1$




✓ $|B[U]|=\mathbf{7}$, $b(U)-1=\mathbf{8}$




× $|B(U)|=\mathbf{6}$, $b(U)-1=\mathbf{5}$

- ➢ (i): Direct sum of **uniform matroids**
- ➢ (ii): **Sparsity matroid**

**Sparsity matroid** **[cf. Frank 11]**

Graph $G=(V,E)$, Vector $\boldsymbol{b} \in \mathbf{Z}^V$, Integer $k \ge 0$

➢ $\{B \subseteq E : |B[U]| \le b(U) - k\}$ is an independent set family of a matroid

✓ $k$ disjoint branchings: $\mathrm{indeg}(u) \le k \quad (u \in V)$

$|B[U]| \le k|U| - k \quad (\emptyset \ne U \subseteq V)$

# Results on *b*-branchings



[Kakimura, Kamiyama, **T.** 18]

**Result (A)**

**TDI linear system** [Schrijver 82]

- Holds for any matroid intersection [Edmonds 70]

**Result (B)**

**Multi-phase greedy algorithm** for max weight *b*-branching

- More tractable than **bipartite matching**

**Result (C)**

**Packing Theorem**

# (A) TDI System for $b$-branchings



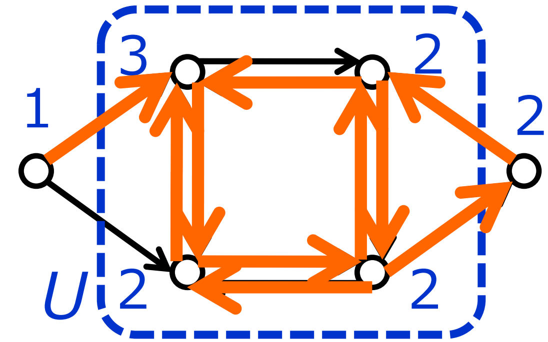
**Linear Program (P)** in variable $x \in \mathbf{R}^A$

max. $\sum w(a)x(a)$

s.t. $x(\delta^{in}(u)) \leq b(u)$ $(u \in V)$

$x(A[U]) \leq b(U) - 1$ $(\emptyset \neq U \subseteq V)$

$x(a) \geq 0$ $(a \in A)$

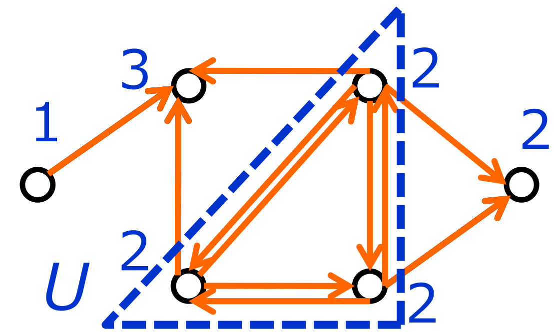



✓ $|B[U]| = \mathbf{7}$, $b(U)-1 = \mathbf{8}$

**TDI Theorem**

**This linear system is TDI**, i.e.,

- (P) has an integer optimal solution
- If $w$ is integer, the dual program also has an integer optimal solution




× $|B(U)| = \mathbf{6}$, $b(U)-1 = \mathbf{5}$

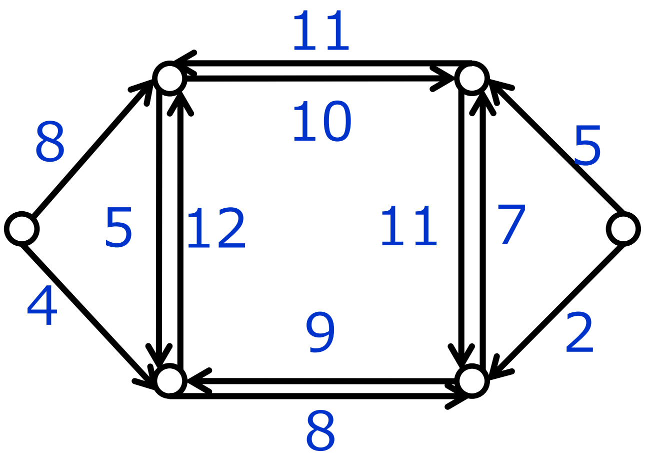
- Holds for any matroid intersection [Edmonds 70]

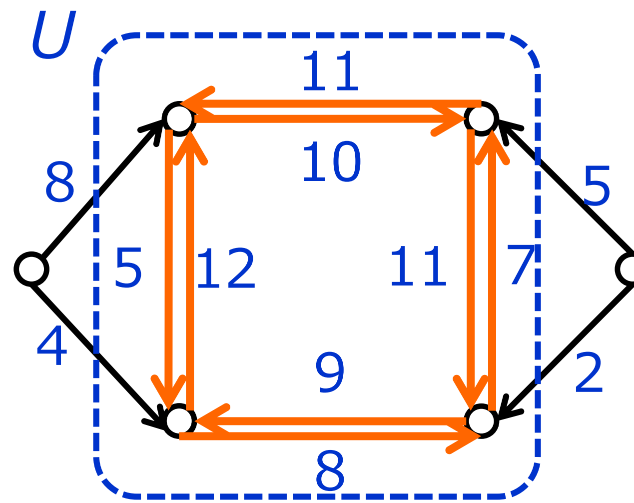
# (B) Multi-phase Greedy Algorithm



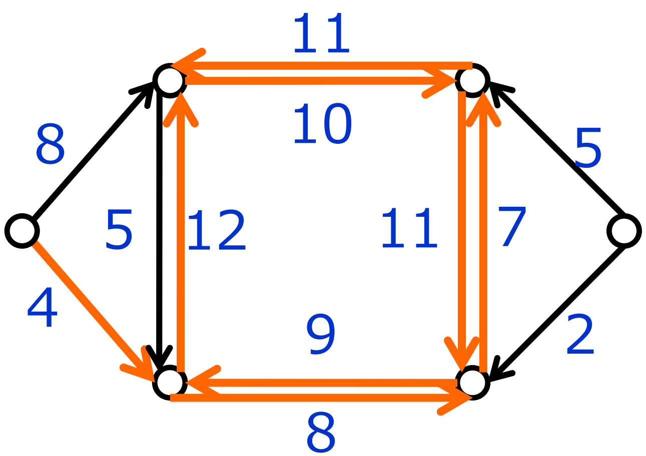
$b(u)=2\ (^{\forall}u \in V)$

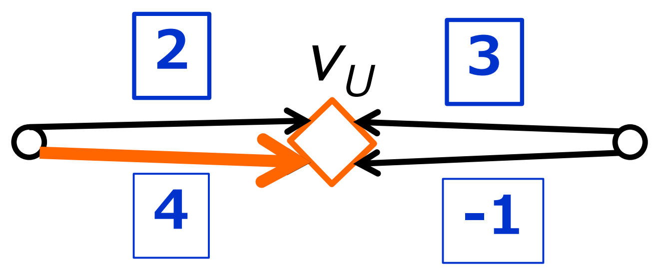
[Kakimura, Kamiyama, **T.** 18]




$|B[U]|=b(U)$




$\boldsymbol{b(v_U) := 1}$

**2** = 8 + 5 − 11
**4** = 4 + 5 − 5
**3** = 5 + 5 − 7
**-1** = 2 + 5 − 8

# (C) Packing Disjoint $b$-branchings

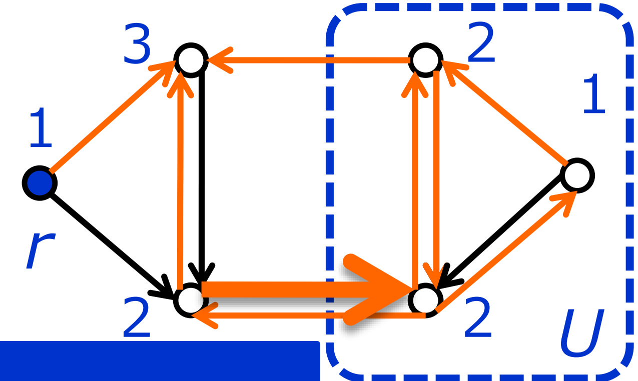


[Kakimura, Kamiyama, **T.** 18]

**Theorem**

$b$-branching with $\mathrm{indeg}(u)=b(u)$ $(u\in V\setminus\{r\})$ $\Leftrightarrow$

- $\boldsymbol{|\delta^{in}(u)| \geq b(u)}$ $(u \in V\setminus\{r\})$
- $\boldsymbol{|\delta^{in}(U)| \geq 1}$ $(\emptyset \neq U \subseteq V\setminus\{r\})$




**Theorem**

***k* disjoint** $b$-branchings with $\mathrm{indeg}(u)=b(u)$ $(u\in V\setminus\{r\})$ $\Leftrightarrow$

- $\boldsymbol{|\delta^{in}(u)| \geq k\cdot b(u)}$ $(u \in V\setminus\{r\})$
- $\boldsymbol{|\delta^{in}(U)| \geq k}$ $(\emptyset \neq U \subseteq V\setminus\{r\})$

**Th.** [Edmonds 67, Bock 71, Fulkerson 74]

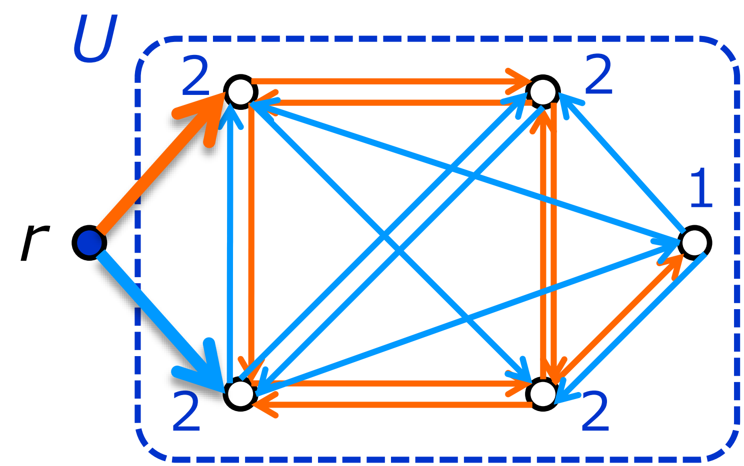
$r$-arborescence

$\Leftrightarrow |\delta^{in}(U)| \geq 1$ $(\emptyset\neq U\subseteq V\setminus\{r\})$

**Th.** [Edmonds 73]

***k* disjoint** $r$-arborescence

$\Leftrightarrow |\delta^{in}(U)| \geq k$ $(\emptyset\neq U\subseteq V\setminus\{r\})$



# Outline



(3) ***b***-bibranching

**Our result**

- TDI system
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(2-2) Bibranching

- TDI system [Schrijver 82]
- Packing [Schrijver 82]
- M-convex submodular flow formulation [**T.** 12]

(2-1) ***b***-branching

- TDI system [Kakimura, Kamiyama, **T.** 18]
- Packing [Kakimura, Kamiyama, **T.** 18]

(1) Branching

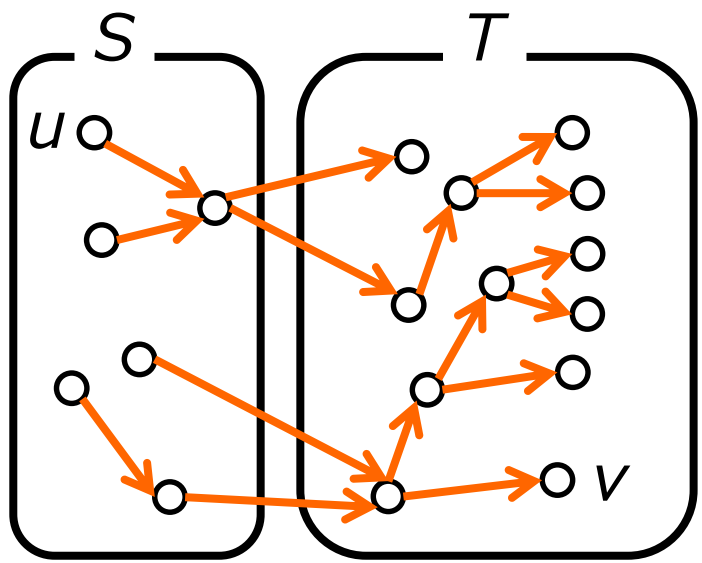
- TDI system [Edmonds 70]
- Packing [Edmonds 73]

# Bibranching

- ◆ Digraph $(V, A)$
- ◆ Partition $\{S,T\}$ of $V$

**Definition** **[Schrijver 82]**

- $B \subseteq A$ is a **bibranching** $\Leftrightarrow$
  In $(V, B)$,
  - ➢ $^{\forall} v \in T$ is reachable form $S$
  - ➢ $^{\forall} u \in S$ reaches $T$




**Bibranching** $B \subseteq A$

- Can assume $A[T, S] = \emptyset$

◆ Motivations:

- ➢ Generalization of **Arborescence** / **Bipartite Edge Cover**
- ➢ Packing theorem is used in a proof of *Woodall's conjecture* in *source-sink connected digraphs*

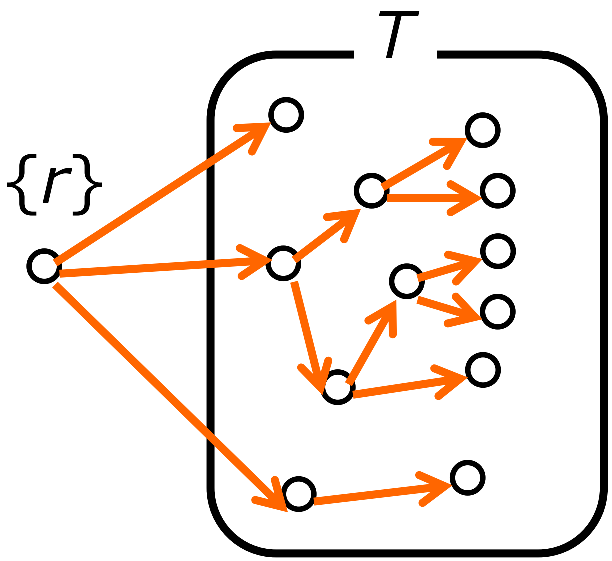
[Schrijver 82]

# Special Cases of Bibranchings



## ◆ Arborescence

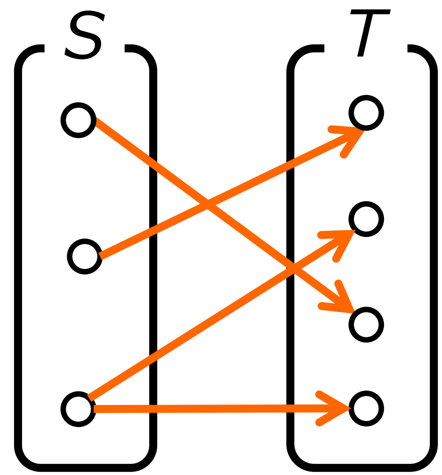
- $S = \{r\}$




Minimal bibranching
= ***r*-aborescence**

## ◆ Bipartite edge cover

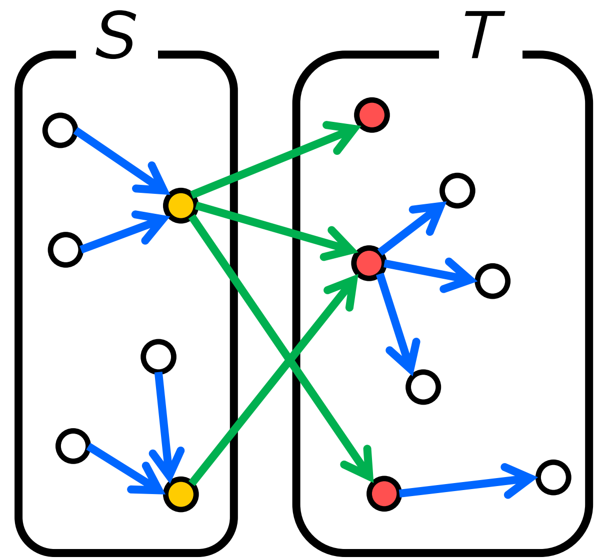
- $A[S] = A[T] = \emptyset$




Bibranching = **Edge cover**

$B \subseteq A$ is a **bibranching**
⇔ In $(V, B)$,
- $^{\forall} v \in T$ is reachable from $S$
- $^{\forall} u \in S$ reaches $T$

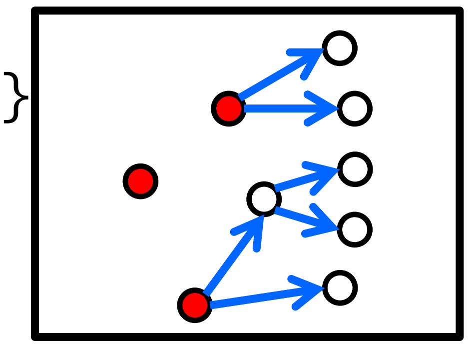
# Alternative Definition of Bibranchings




## Alternative Definition of Bibranchings

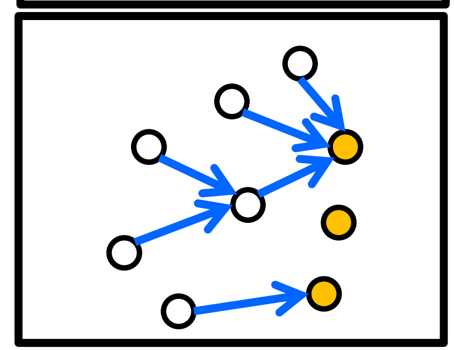
$B \subseteq A$ is a **bibranching** $\Leftrightarrow$
For $F = B[S,T]$,
- $B[T]$ is a *branching* with $R(B[T]) = \partial^- F$
- $B[S]$ is a *cobranching* with $R^*(B[S]) = \partial^+ F$

- $B \subseteq A$ is a ***cobranching*** $\Leftrightarrow$ Reversal of a branching
- **Root $R(B)$** of a branching $B = \{v \in V\colon |B \cap \delta^{\mathrm{in}} v| = 0\}$
- **Root $R^*(B)$** of a cobranching $B = \{v \in V\colon |B \cap \delta^{+\mathrm{out}} v| = 0\}$



$B \subseteq A$ is a **bibranching**
$\Leftrightarrow$ In $(V, B)$,
- $^\forall t \in T$ is reachable from $S$
- $^\forall s \in S$ reaches $T$

Minimal ones



# Results on Bibranchings



## Results on Bibranchings

**(A)** **TDI linear system** [Schrijver 82]

- **NOT** follows from TDIness of matroid intersection

**(B)** Algo. for min-weight bibranching [Keijsper, Pendavingh 98]

- **NOT greedy**, but as fast as bipartite edge cover algorithm

**(C)** **Packing Theorem** [Schrijver 82]

- Used in a proof for *Woodall's conjecture* for source-sink connected digraphs [Schrijver 82]

**(D)** **$M^\natural$-convex submodular flow formulation** [**T.** 12]

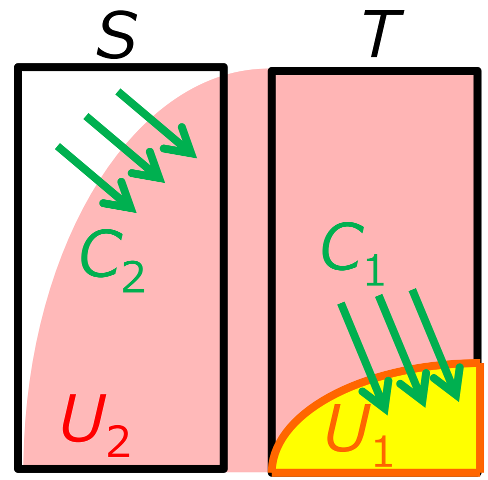
**(E)** (D) is obtained from (A) by *Benders decomposition*

[Murota, **T.** 17]

# (A) TDI System for Bibranching 

**Definition** **[Schrijver 82]**

- $C \subseteq A$ is a **bicut** $\Leftrightarrow C = \delta^{in}(U)$, where $\emptyset \neq U \subseteq T$ or $T \subseteq U \subsetneq V$




**Linear Program (P)** in variable $x \in \mathbf{R}^A$

min. $\sum w(a)x(a)$

s.t. $x(C) \geq 1$ (C: bicut)

$x(a) \geq 0$ $(a \in A)$

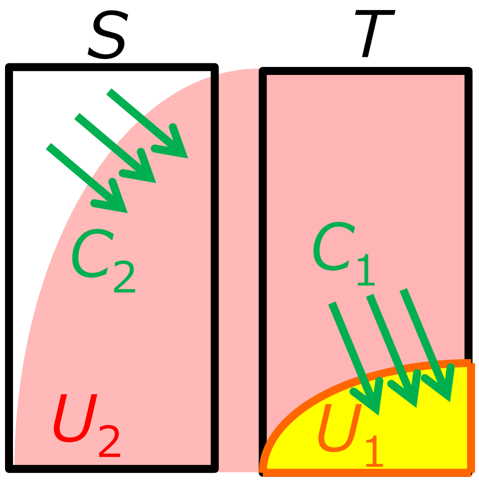
**TDI theorem** [Schrijver 82]

This linear system is TDI

$B \subseteq A$ is a **bibranching**
$\Leftrightarrow$ In $(V, B)$,
- $^\forall v \in T$ is reachable from $S$
- $^\forall u \in S$ reaches $T$

# (C) Packing Disjoint Bibranchings






**Disjoint Bibranchings Theorem** [Schrijver 82]

$D$ has ***k* arc-disjoint** bibranchings
$\Leftrightarrow$ $|C| \geq k$  ($C$: bicut)

- Can be proved by *supermodular colouring* [Schrijver 85][Tardos 85]

- Applied to a proof for *Woodall's conjecture* for *source-sink connected digraphs* [Schrijver 82]

**Woodall's conjecture** [1978]

Every directed cut has $\geq k$ arcs
$\Rightarrow$ $^{\exists}k$ disjoint directed-cut covers (dijoins)

# Outline



**(3) *b*-bibranching**

**Our result**

- TDI system
- Packing
- M-convex submodular flow formulation

**(2-2) Bibranching**

- TDI system [Schrijver 82]
- Packing [Schrijver 82]
- M-convex submodular flow formulation [**T.** 12]

**(2-1) *b*-branching**

- TDI system [Kakimura, Kamiyama, **T.** 18]
- Packing [Kakimura, Kamiyama, **T.** 18]

**(1) Branching**

- TDI system [Edmonds 70]
- Packing [Edmonds 73]

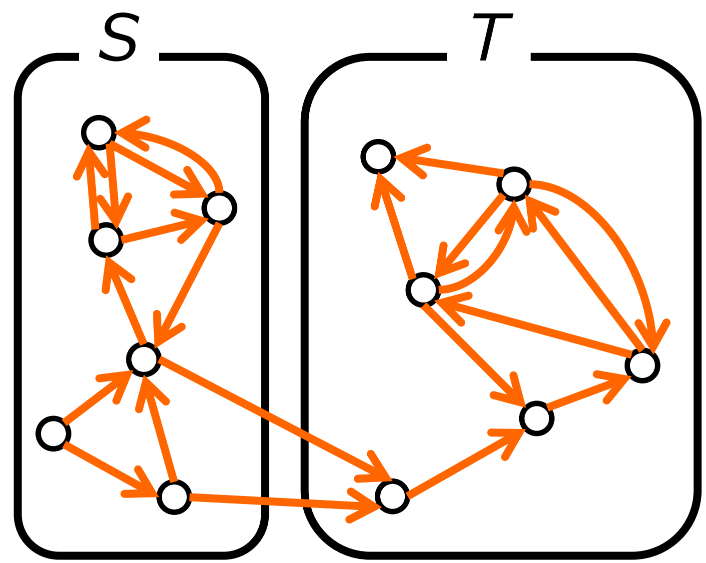
# *b*-bibranching



- Digraph $(V, A)$
- Positive integer vector $\boldsymbol{b} \in \mathbf{Z}^V$ on $V$
- Partition $\{S, T\}$ of $V$

**Definition**

- $B \subseteq A$ is a ***b*-bibranching** $\Leftrightarrow$
  In $(V, B)$,
  - $\mathrm{indeg}(v) \geq b(v) \quad (v \in T)$
  - $\mathrm{outdeg}(u) \geq b(u) \ (u \in S)$
  - $^{\forall} v \in T$ is reachable from $S$
  - $^{\forall} u \in S$ reaches $T$




***b*-bibranching $B \subseteq A$**

$(b(u) \equiv 2)$

- $b(u) \equiv 1$ ➔ Bibranching
- $S = \{r\}$ ➔ ***b*-branching** (Mininal ***b*-bibranchings**)

# Our Results on *b*-bibranchings



## Result (A)

**TDI linear system**

- Does NOT follow from TDIness of matroid intersection
- Implies the poly.-time solvability by *Ellipsoid method*

## Result (C)

**Packing Theorem**

- Proved by Packing ***b***-branchings [Kakimura, Kamiyama, **T.** 18] + Supermodular colouring [Schrijver 85][Tardos 85]

## Result (D)

**M$^\natural$-convex submodular flow formulation**

- Implies a *combinatorial algorithm* [Iwata, Shigeno 02] [Iwata, Moriguchi, Murota 05]

# TDI system for *b*-bibranchings

**Linear Program (P) in variable $x \in \mathbf{R}^A$**

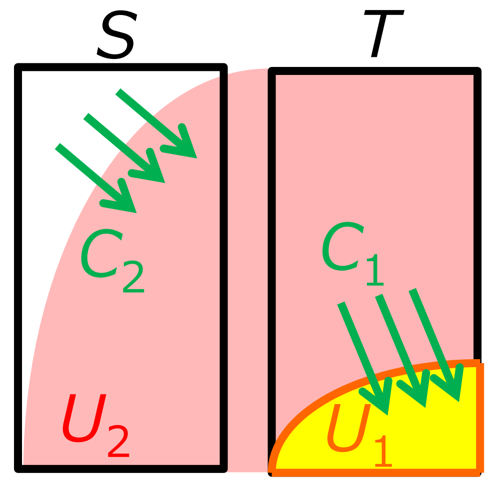
max. $\sum w(a)x(a)$

s.t. $x(\delta^{in}(v)) \geq b(v) \quad (v \in T)$

$x(\delta^{out}(u)) \geq b(u) \quad (u \in S)$

$x(C) \geq 1 \quad$ ($C$: bicut)

$x(a) \geq 0 \quad (a \in A)$




**TDI Theorem** [Our Result]

This linear system is **TDI**.

**Corollary** [Our Result]

This linear system determines the ***b*-bibranching polytope**

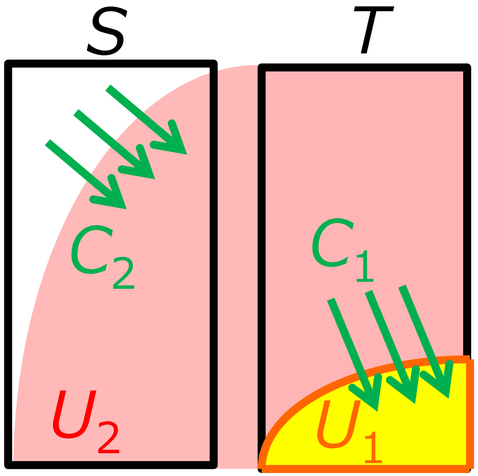
# Packing disjoint *b*-bibranchings



**Disjoint *b*-bibranchings theorem** [Our Result]

$D$ has ***k* arc-disjoint *b*-bibranchings** ⇔

- $\mathrm{indeg}(v) \geq k \cdot b(v)$ $(v \in T)$
- $\mathrm{outdeg}(u) \geq k \cdot b(u)$ $(u \in S)$
- $|C| \geq k$ ($C$: bicut)



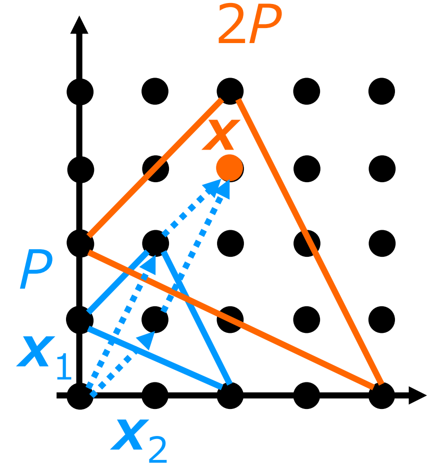

**Recall: Disjoint Bibranchings Theorem** [Schrijver 82]

$D$ has ***k* arc-disjoint** bibranchings
⇔ $|C| \geq k$ ($C$: bicut)

- Implies the **integer decomposition property** of the ***b***-bibranching polytope

**Integer decomposition property of *P***

$\forall k \in \mathbf{Z}_{++}$, $\forall \boldsymbol{x} \in kP \cap \mathbf{Z}^A$, $\boldsymbol{x} = \boldsymbol{x}_1 + \cdots + \boldsymbol{x}_k$ $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in P \cap \mathbf{Z}^A)$



# Outline

(3) ***b***-bibranching

**Our result**

- TDI system
- Packing
- M-convex submodular flow formulation

(2-2) Bibranching

- TDI system [Schrijver 82]
- Packing [Schrijver 82]
- M-convex submodular flow formulation [**T.** 12]

(2-1) ***b***-branching

- TDI system [Kakimura, Kamiyama, **T.** 18]
- Packing [Kakimura, Kamiyama, **T.** 18]

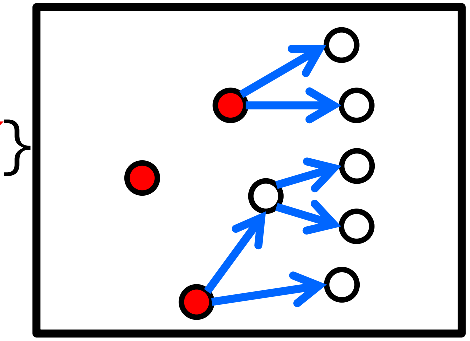
(1) Branching

- TDI system [Edmonds 70]
- Packing [Edmonds 73]

# M♮-convexity of shortest branchings 

Define $f_T$: $2^T \to \mathbf{Z}\cup\{+\infty\}$ by

$$f_T(X) = \begin{cases} \min\{w(B) \mid B\text{: branching } D[T],\ R(B) = X\} \\ +\infty \text{ if no such } B \text{ exists} \end{cases}$$



**Theorem [T. 11]**

$f_T$ is an M♮-convex function on $\{0,1\}^T$

- Can be proved by disjoint branchings theorem [Edmonds 73]

**Definition [Murota, Shioura 99]**

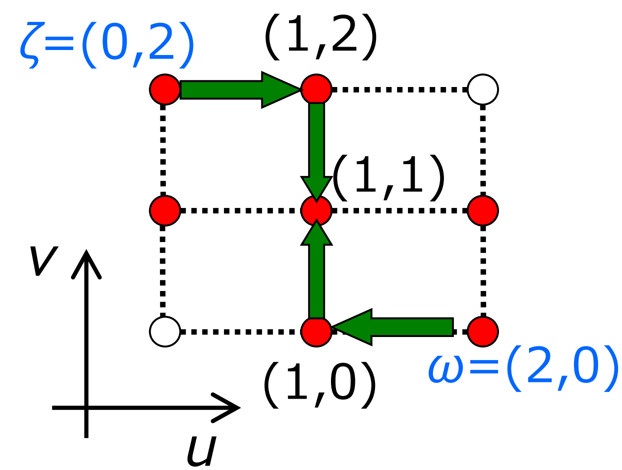
$f$: $\mathbf{Z}^V \to \mathbf{Z}\cup\{+\infty\}$ is an **M♮-convex function**

$\Leftrightarrow$ $\forall \omega, \zeta \in \mathbf{Z}^V$, $\forall u \in \mathrm{supp}^+(\omega - \zeta)$,

$$f(\omega)+f(\zeta) \geq f(\omega - \chi_u) + f(\zeta + \chi_u)$$

or $\exists v \in \mathrm{supp}^-(\omega - \zeta)$,

$$f(\omega)+f(\zeta) \geq f(\omega - \chi_u + \chi_v) + f(\zeta + \chi_u - \chi_v)$$



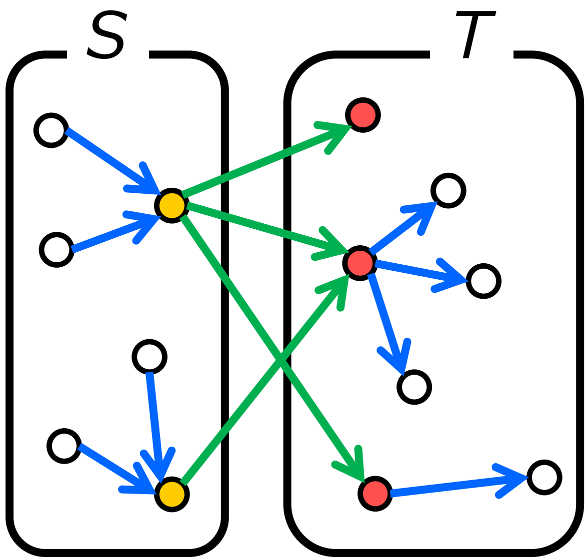
# M♮-conv. Submod. Flow for bibranching



## M♮SF for Shortest Bibranching [T. 12]

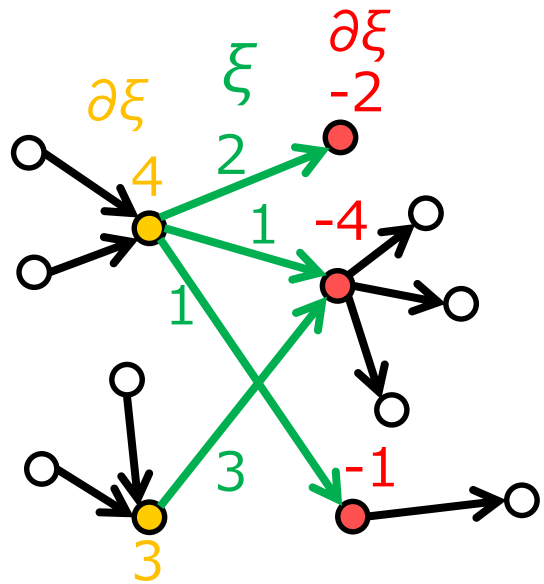
min. $w(F) + f_S(\partial^+ F) + f_T(\partial^- F)$
s.t. $F \subseteq A[S,T]$

$$f_T(X) = \begin{cases} \min\{w(B) \mid B\text{: branching in } G[T],\ R(B) = X\} \\ +\infty \text{ if no such } B \text{ exists} \end{cases}$$




## Theorem [T. 12]

$f_T$: M♮-convex function on $\{0,1\}^T$ is extendable to $\mathbf{Z}^T$ maintaining M♮-convexity




## M♮-convex Submodular Flow [Murota 99]

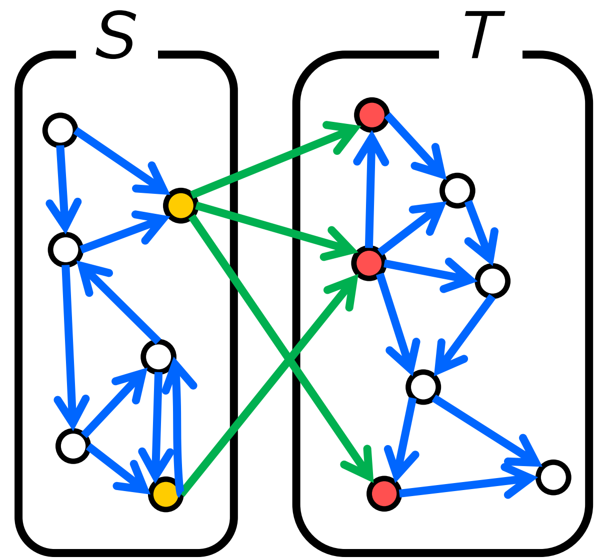
min. $w(\xi) + f(\partial\xi)$
s.t. $l \leq \xi \leq u$
$\xi \in \mathbf{Z}^A$

$f$: M♮-convex

# M$^\natural$-conv. Submod. Flow for *b*-bibranching

## M$^\natural$SF for shortest *b*-bibranching

min. $w(F) + g_S(\partial^+ F) + g_T(\partial^- F)$
s.t. $F \subseteq A[S,T]$




$$f_T(\boldsymbol{x}) = \begin{cases} \min\{w(B) \mid B\text{: } \boldsymbol{b}\text{-branching } G[T],\ \boldsymbol{x}+\partial^- B \geq \boldsymbol{b}\} \\ +\infty \quad \text{if no such } B \text{ exists} \end{cases}$$

## Theorem [Our result]

$g_T$: M$^\natural$-convex function $\mathbf{Z}^T$

- Can be proved by disjoint b-branchings theorem [Kakimura, Kamiyama, T. 18]

## Corollary [Our result]

The shortest ***b***-bibranching problem can be solved in polynomial time.

- Combinatorial algorithms for M$^\natural$-convex submodular flow [Iwata, Shigeno 02] [Iwata, Moriguchi, Murota 05]

# Summary and Future Work



***b*-bibranching**

**Our result**

- TDI system
- Packing
- M-convex submodular flow formulation

**Bibranching**

- TDI system [Schrijver 82]
- Packing [Schrijver 82]
- M-convex submodular flow formulation [**T.** 12]

***b*-branching**

- TDI system [Kakimura, Kamiyama, **T.** 18]
- Packing [Kakimura, Kamiyama, **T.** 18]

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- Direct combinatorial algorithm
  - Like [Keijsper, Pendavingh 98] for bibranchings
- **Application**
  - **Theoretical**: Like Woodall's conjecture [Schrijver 82]
  - **Practical**: Evacuation/Communication Network Design

# END of Slides