Excluded *t*-factors in Bipartite Graphs:

A Unified Framework for

- Nonbipartite Matchings and Restricted 2-matchings
- Blossom and Subtour Elimination Constraints

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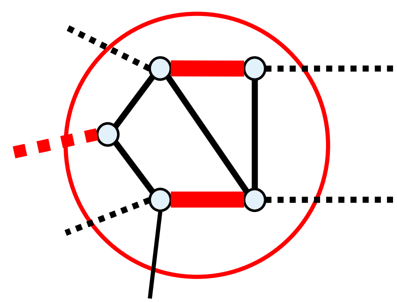
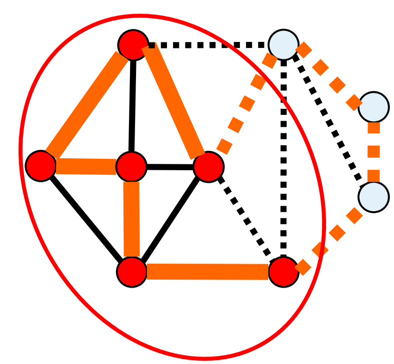
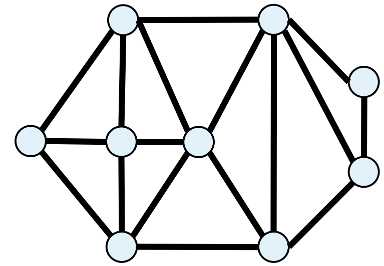
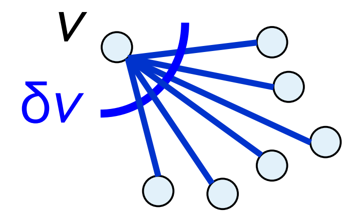




Matching, 2-matching, and t-matching

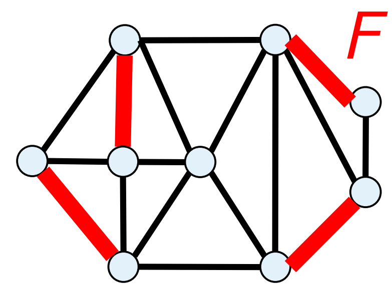
- $G = (V,E)$: Simple, Undirected

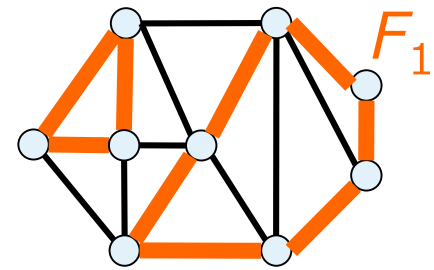


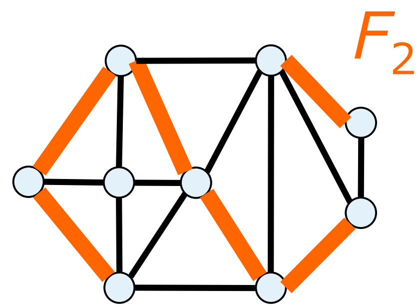


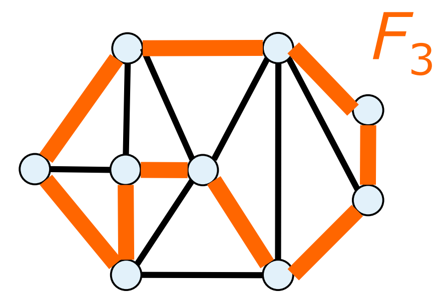
Definition

- $F \subseteq E$ is a **matching** $\Leftrightarrow |F \cap \delta v| \leq 1 \quad {}^\forall v \in V$
- $F \subseteq E$ is a **2-matching** $\Leftrightarrow |F \cap \delta v| \leq 2 \quad {}^\forall v \in V$
- $F \subseteq E$ is a ***t*-matching** $\Leftrightarrow |F \cap \delta v| \leq t \quad {}^\forall v \in V$



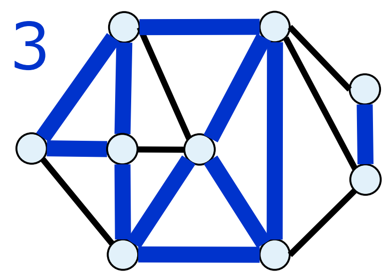






- **Just keep t=1,2 in mind**
- No theoretical difference in ${}^\forall t \in \mathbf{Z}_{>0}$

$t = 3$



Our Framework : What are contained ?

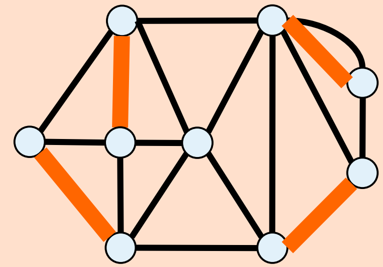
Our Framework

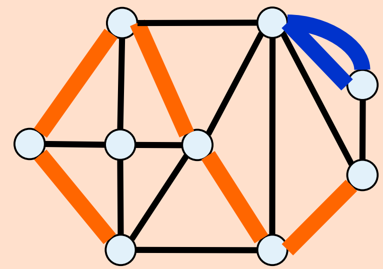
- **Matching**

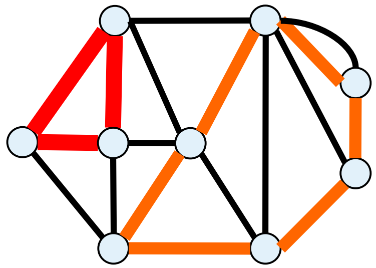
Restriction

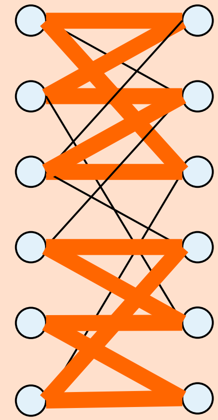
- **Triangle-free 2-matching** with edge-multiplicity
- **Square-free 2-matching in bipartite graph**

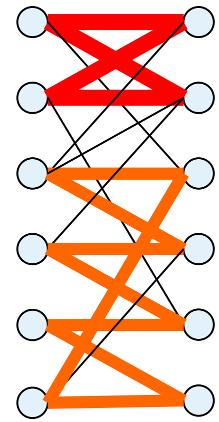
- **Hamilton cycle**

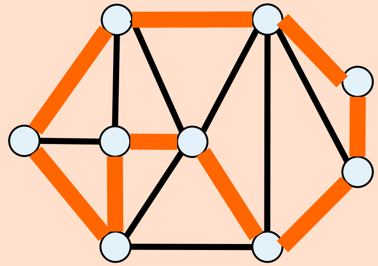












Our Result : What did we solve ?

Our Framework

- **Matching**
- **Triangle-free 2-matching** with edge-multiplicity
- **Square-free 2-matching in bipartite graph**

↑ **P**

↓ **NP-hard**

- **Hamilton cycle**

Our Result

- ➢ ***Min-max theorem***
- ➢ ***LP with dual integrality***
- ➢ ***Combinatorial algorithm***

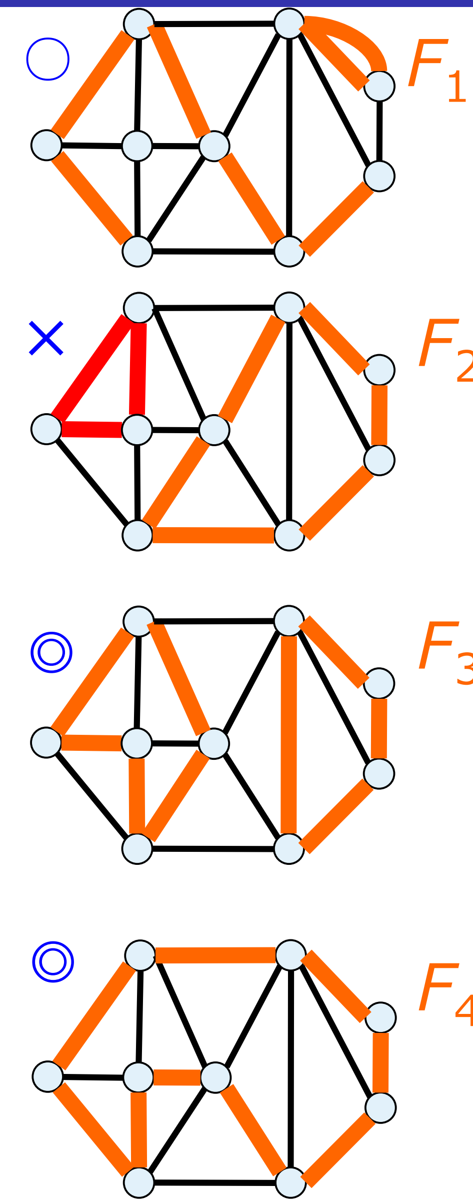
- **Even factor** [Cunningham, Geelen '01]
- **$K_{t,t}$-free t-matching** [Frank '03]
- **2-matching covering 3,4-edge cuts** [Kaiser, Škrekovski '04,08] [Boyd, Iwata, **T.** '13]

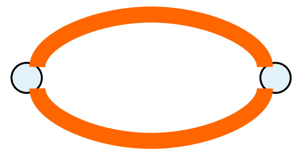
Contents

Triangle-free 2-matching

Definition (Triangle-free 2-matching)

- 2-matching $\boldsymbol{x} \boldsymbol{\in} \{\mathbf{0,1,2}\}^E$ is **Triangle-free**
  $\Leftrightarrow$ Excluding cycles of length 3



➢ Allowing multiplicity 2: 

Theorem [Cornuéjols & Pulleyblank '80]

- Max. $\sum x(e)$: **P**
- Max. $\sum w(e)x(e)$: **P**
  - ➢ *Min-max theorem*
  - ➢ *LP with dual integrality*
  - ➢ *Combinatorial algorithm*

- No multiplicity allowed:
  - ➢ Max. $|F|$: Algorithm [Hartvigsen '84]
  - ➢ Max. $w(F)$: **Open**
    - ➢ *Discrete convexity* [Kobayashi '14]

Square-free 2-matching in bipartite graph

Definition (Square-free 2-matching)

- 2-matching $F \subseteq E$ is **Square-free**
  $\Leftrightarrow$ Excluding cycles of length 4

Previous work for bipartite graphs

- Max. $|F|$: **P**
  - *Min-max theorem* [Z. Király '99, Frank '03]
  - *Combinatorial algorithm* [Hartvigsen '06; Pap '07]
  - *Canonical decomposition* [**T.** '15]
- Max. $w(F)$: **NP-hard** [Z. Király '99]
  - **P** under *a certain assumption on **w*** (☞p.16)
    - ✓ *LP with dual integrality* [Makai '07]
    - ✓ *Combinatorial algorithm* [**T.** '09]

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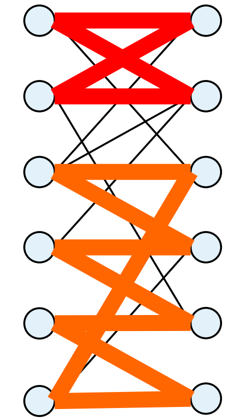

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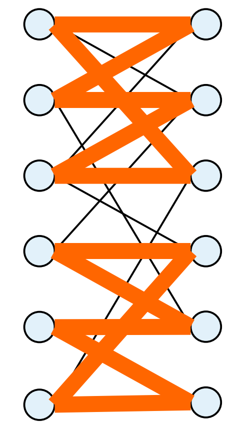
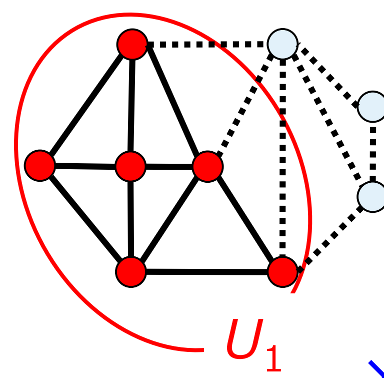


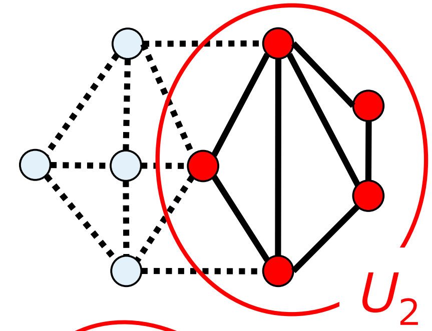
- Max. $|F|$ in **nonbipartite** graphs: **Open**
  - *Discrete convexity* [Kobayashi, Szabó, **T.** '12]

Contents

Our Framework: $\mathcal{U}$-feasible t-matching

- **$\mathcal{U} \subseteq 2^V$: Vertex subset family**



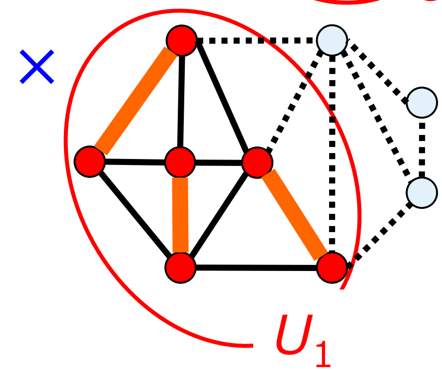


Definition

t-matching $F \subseteq E$ is **$\mathcal{U}$-feasible**

$\Leftrightarrow$ $|F[U]| \le \left\lfloor \frac{t|U|-1}{2} \right\rfloor$ ${}^{\forall} U \in \mathcal{U}$

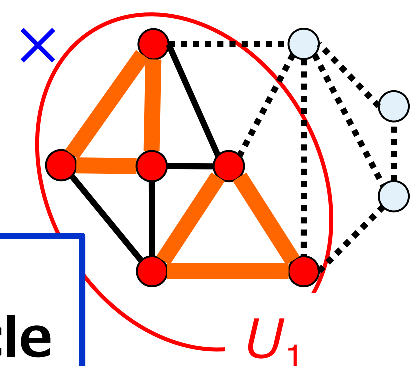
$\Leftrightarrow$ Excluding t-factors in $G[U]$ ${}^{\forall} U \in \mathcal{U}$

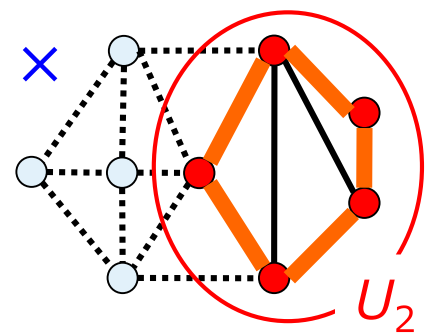


- $t=1$: $|F[U]| \le \left\lfloor \frac{|U|-1}{2} \right\rfloor = \begin{cases} \frac{|U|}{2} - 1 & (|U|\text{: even}) \\ \cancel{\frac{|U|-1}{2} \quad (|U|\text{: odd})} \end{cases}$

- $t=2$: $|F[U]| \le \left\lfloor \frac{2|U|-1}{2} \right\rfloor = |U| - 1$

[**T.** '16]





➢ **$\mathcal{U} = 2^V \setminus \{\emptyset, V\}$**

➔ $\mathcal{U}$-feasible 2-factor = **Hamilton cycle**

Our Result

Our assumption

◆ G: **Bipartite**

◆ $^\forall U \in \mathcal{U}$ is **"factor-critical"** (☞ p. 13)

Our result

➢ ***Min-max theorem***

➢ ***Combinatorial algorithm***

Weighted (Assumption on *w*)

➢ ***LP with dual integrality***

➢ ***Combinatorial algorithm***

◆ Problems accepting this assumption:

➢ **Square-free 2-matching**

- $\mathcal{U} = \{U : U \subseteq V, |U|=4\}$
- $t=2$

➢ **Nonbipartite matching**

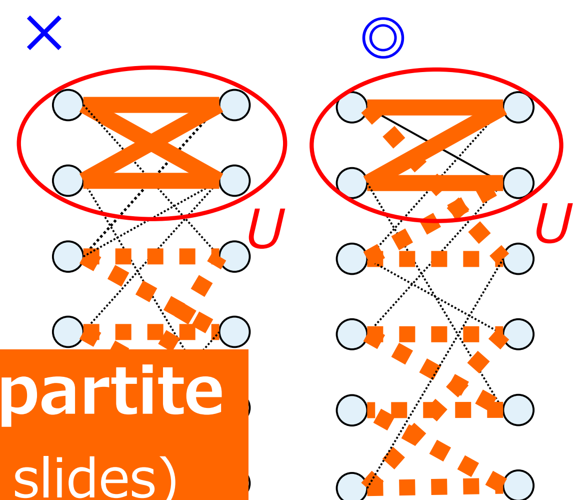
➢ **Triangle-free 2-matching**

➢ **Even factor**

➢ **$K_{t,t}$-free t-matching**

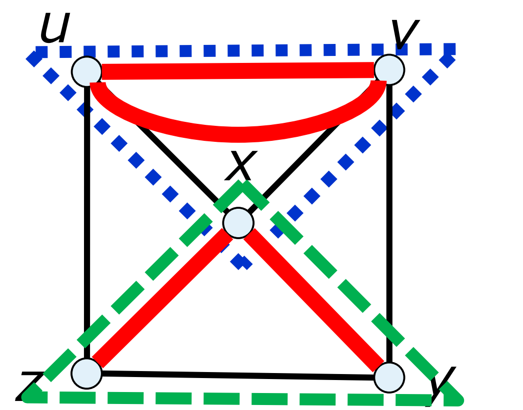
Nonbipartite

(☞Next slides)

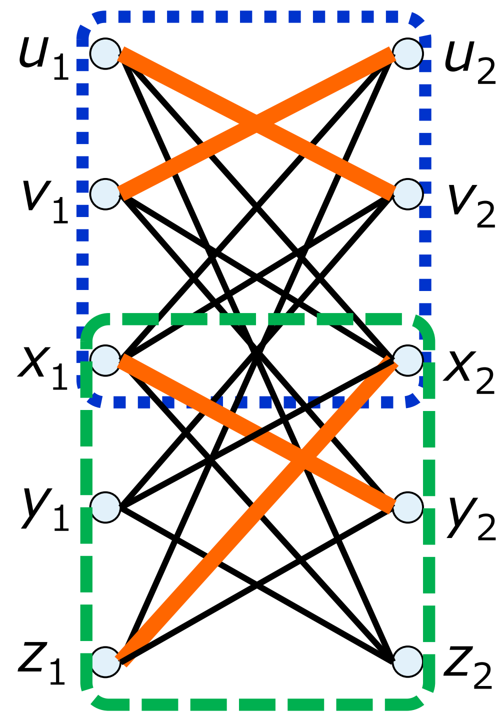


Special Case: Triangle-free 2-matching

- $G=(V,E)$: **Nonbipartite graph**



- $G'=(V',E')$: **Bipartite graph**
  - $V' = V_1 \cup V_2$
  - $E' = \{u_1v_2, v_1u_2 : uv \in E\}$
- $t = 1$
- $\mathcal{U} = \{U_1 \cup U_2 : U \subseteq V, |U|=3\}$

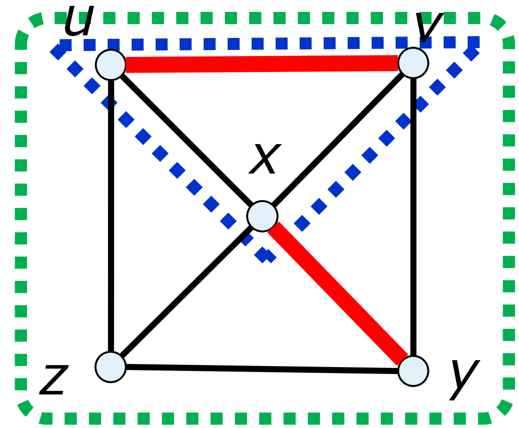


Proposition

|max. triangle-free 2-matching in G|
= |max. $\mathcal{U}$-feasible 1-matching in G'|

Special Case: Nonbipartite Matching

- $G=(V,E)$: **Nonbipartite graph**



- $G'=(V',E')$: **Bipartite graph**
  - $V' = V_1 \cup V_2$
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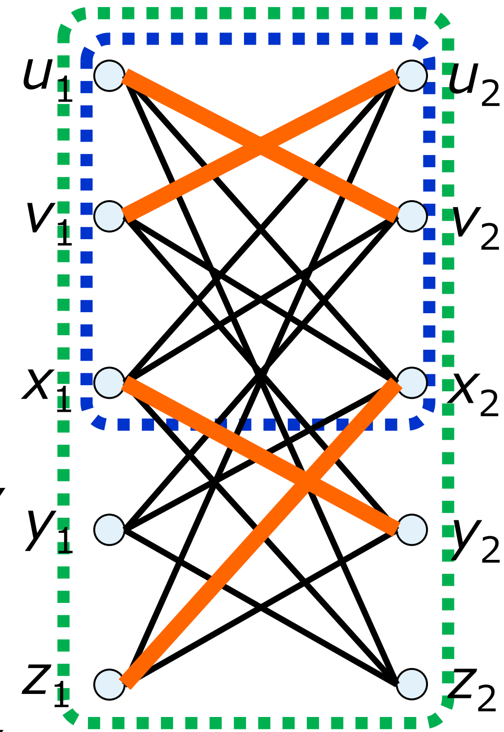
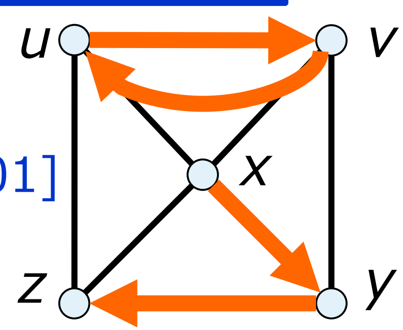
- $t = 1$
- $\mathcal{U} = \{U_1 \cup U_2 : U \subseteq V,\ |U| \text{ is odd}\}$



Proposition

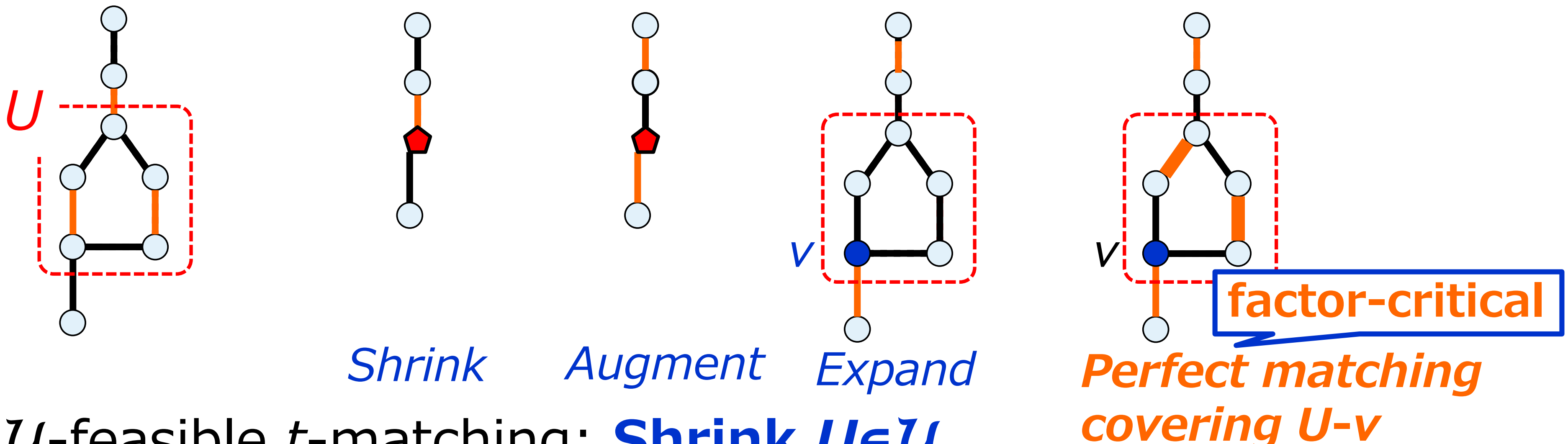
$2 \cdot |\text{max matching in } G|$
$= |\text{max } \mathcal{U}\text{-feasible 1-matching in } G'|$

Dipaths and even dicycles
= Even factor [Cunningham, Geelen '01]

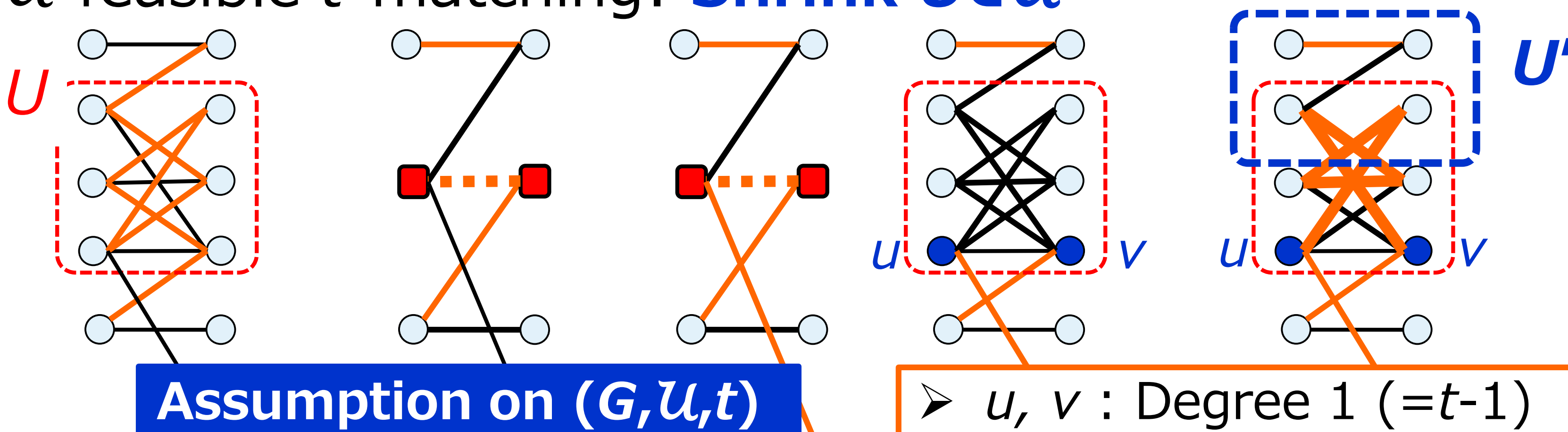


Algorithm + Factor-criticality of $U \in \mathcal{U}$

- Nonbipartite matching: **Shrink odd cycles** [Edmonds '65]



- $\mathcal{U}$-feasible t-matching: **Shrink $U \in \mathcal{U}$**



Assumption on $(G,\mathcal{U},t)$

$U \in \mathcal{U}$ is **"factor-critical"**

- u, v : Degree 1 $(=t-1)$
- $U-\{u,v\}$: Degree 2 $(=t)$
- Feasible for $U' \in \mathcal{U}$ crossing U

Min-max Theorem

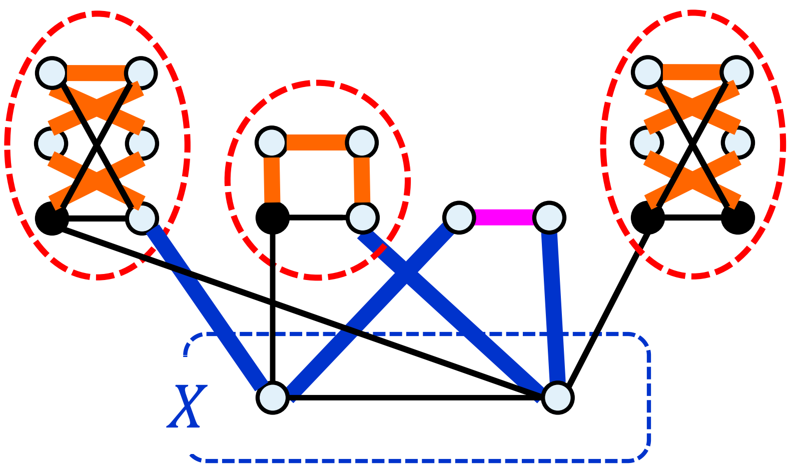
Theorem

- G: Bipartite
- $^{\forall}U \in \mathcal{U}$ is "factor-critical"

- Nonbipartite matching
- Triangle-free 2-matching
- Square-free 2-matching
- Even factor
- $K_{t,t}$-free t-matching

➔ max{|F| : F is a $\mathcal{U}$-feasible t-matching}

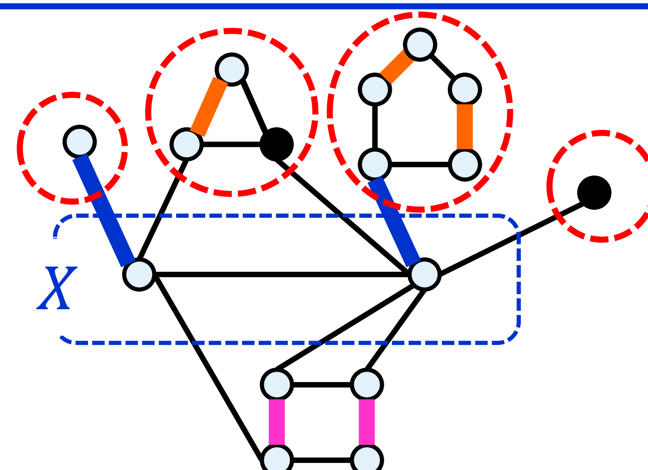
$$=\min\{\, t|X| + |E[C_{V-X}]| + \sum_{U \in \mathcal{U}(V-X)} \left\lfloor \frac{t|U|-1}{2} \right\rfloor \,\}$$



Theorem [Tutte '47, Berge '58]

max{|M| : M is a matching}

$$= \frac{1}{2}\min\{|V| + |X| - \mathrm{odd}(X) : X \subseteq V\}$$



Contents

LP for Square-free 2-matching

Max weight square-free 2-matching

Maximize $\quad \Sigma_{e\in E}\, w(e)\, x(e)$

subject to $\quad \Sigma_{e\in\delta v}\, x(e) \leq 2 \qquad (v\in V)$

$\qquad\qquad \boldsymbol{\Sigma_{e\in E[U]}\, x(e) \leq} \color{red}{\mathbf{3}} \qquad (U\subseteq V,\ |U|=4)$

$\qquad\qquad \mathbf{0 \leq x(e) \leq 1} \qquad$ (e∈

$$\left\lfloor \frac{t|U|-1}{2} \right\rfloor$$

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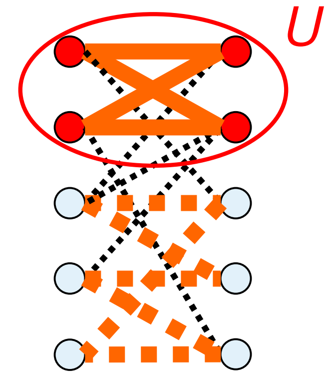

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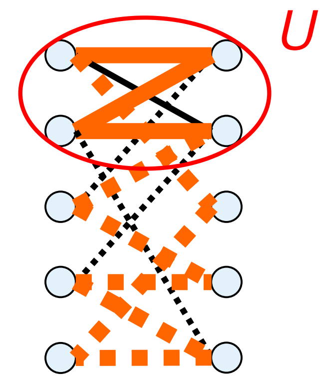


Theorem [Makai '07, T. '09]

Assumption on *w*

- G: Bipartite
- *w* is **vertex-induced on** $^\forall$**square *U***
  i.e., $w(u_1v_1)+w(u_2v_2) = w(u_1v_2)+w(u_2v_1)$

➔ ***This LP** has an **integral opt solution***
The dual LP** has an **integral opt solution

Our Result: LP for $\mathcal{U}$-feasible t-matching

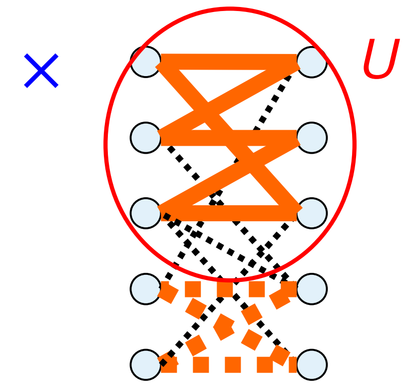
Max weight $\mathcal{U}$-feasible t-matching

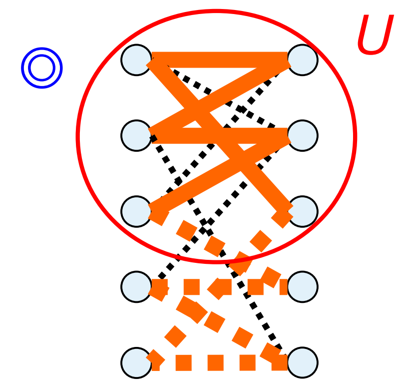
Maximize $\quad \sum_{e\in E} w(e)\, x(e)$

subject to $\quad \sum_{e\in\delta v} x(e) \le t \qquad (v\in V)$

$$\sum_{e\in E[U]} x(e) \le \left\lfloor \frac{t|U|-1}{2} \right\rfloor \qquad (U\in\mathcal{U})$$

$$x(e) \ge 0 \qquad (e\in E)$$



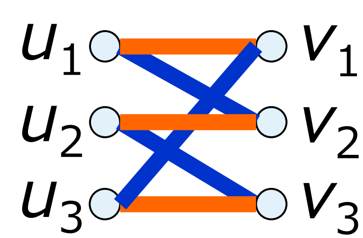


Theorem

Proved by our primal-dual algorithm

- G: Bipartite
- $^{\forall}U\in\mathcal{U}$ is "factor-critical"
- w is **vertex-induced on $^{\forall}U\in\mathcal{U}$**
  i.e., in $G[U]$, the weights of perfect matchings are identical

➔ ***This LP** has an **integral opt solution***
The dual LP** has an **integral opt solution



Our Result: LP for $\mathcal{U}$-feasible t-matching

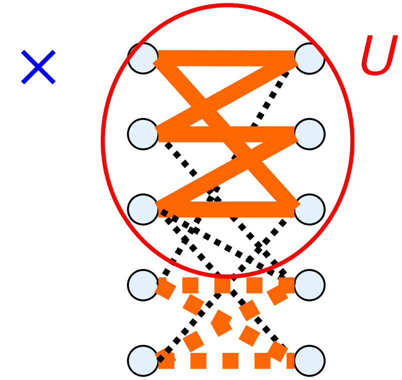
Max weight $\mathcal{U}$-feasible t-matching

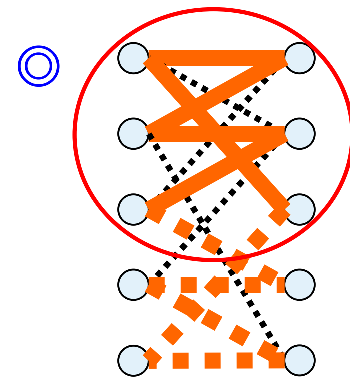
Maximize $\quad \sum_{e\in E} w(e)\, x(e)$

subject to $\quad \sum_{e\in\delta v} x(e) \le t \qquad (v\in V)$

$$\boldsymbol{\sum_{e\in E[U]} x(e)} \le \left\lfloor \frac{t|U|-1}{2} \right\rfloor \qquad (U\in\mathcal{U})$$

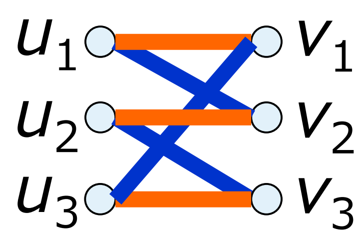
$$\boldsymbol{x(e) \ge 0} \qquad (e\in E)$$





Special cases

- **Subtour Elimination Const.** for TSP
  - ➢ $t=2$ ➔ $\left\lfloor \frac{t|U|-1}{2} \right\rfloor = |U| - 1$
- **Blossom Const.** for matching
  - ➢ $t=1$, $|U|=2\cdot(\text{odd})$ ➔ $\left\lfloor \frac{t|U|-1}{2} \right\rfloor = \left\lfloor \frac{|U|-1}{2} \right\rfloor$

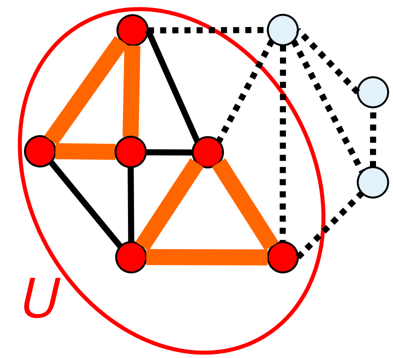


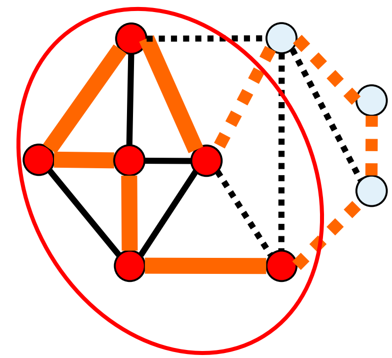
Subtour Elimination for TSP

IP for TSP [Dantzig, Fulkerson, Johnson '54]

Minimize	$\sum_{e\in E} w(e)\, x(e)$	
subject to	$\sum_{e\in\delta v} x(e) = 2$	$(v\in V)$
	$\boldsymbol{\sum_{e\in E[U]} x(e) \le \vert U\vert - 1}$	$(U\subseteq V)$
	$\boldsymbol{x(e)\in\{0,1\}}$	$(e\in E)$

$$\left\lfloor \frac{2|U|-1}{2} \right\rfloor$$





Conjecture [Goemans '95]

w is metric ➔ **Integrality gap** $\le \frac{4}{3}$

i.e., OPT(IP) $\le \frac{4}{3}$ OPT(LP)

Blossom Const for Nonbipartite Matching

Max. weight matching

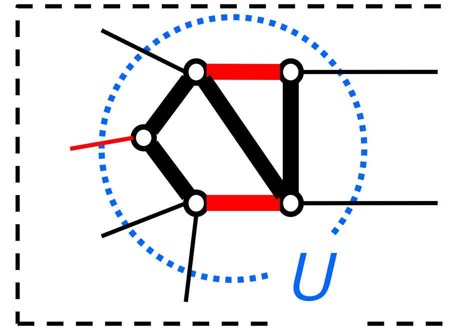
Maximize $\quad \Sigma_{e\in E}\, w(e)\, x(e)$

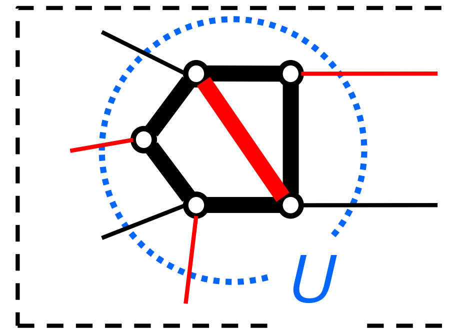
subject to $\quad \Sigma_{e\in \delta v}\, x(e) \le 1 \qquad (v\in V)$

$$\boldsymbol{\Sigma_{e\in E[U]}\, x(e)} \le \frac{|U|-1}{2} \qquad (U\subseteq V,\ |U| \text{ is odd})$$

$$\left\lfloor \frac{|U|-1}{2} \right\rfloor$$

$x(e) \ge 0 \qquad$ E)





Theorem [Cunningham, Marsh '78]

- ***This LP*** *has an* ***integral optimal solution***
- ***The dual LP*** *has an* ***integral optimal solution***

LP for Triangle-free 2-matching

Max weight triangle-free 2-matching

Maximimize $\quad \Sigma_{e\in E}\, w(e)\, x(e)$

subject to $\quad \Sigma_{e\in \delta v}\, x(e) \le 2 \qquad (v\in V)$

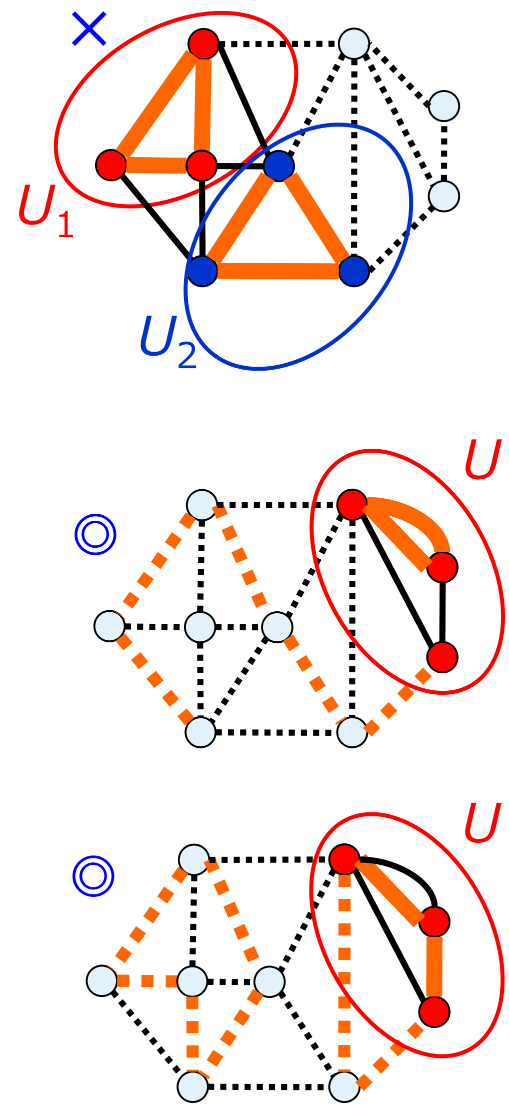
$\quad \boldsymbol{\Sigma_{e\in E[U]}\, x(e)} \le \color{red}{\mathbf{2}} \qquad (U\subseteq V,\ |U|=3)$

$\quad \boldsymbol{x(e)\ge 0} \qquad (e\in E)$

$$\left\lfloor \frac{2|U|-1}{2} \right\rfloor$$

Theorem [Cornuéjols & Pulleyblank ’80]

This LP *has an* ***integer optimal solution***



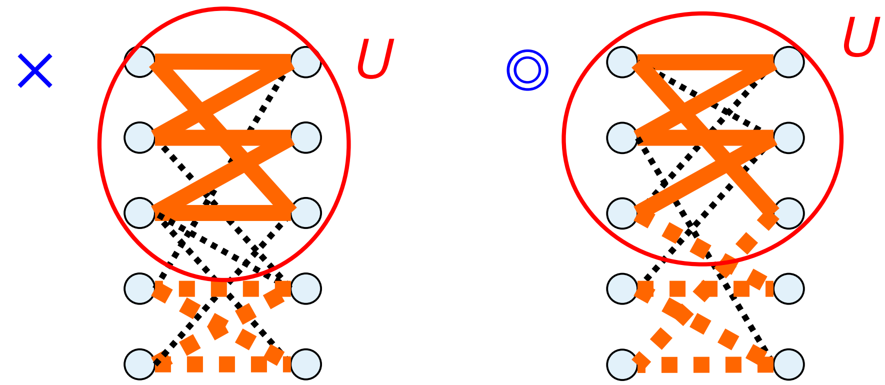
Contents

Summary

Our Framework

- $\mathcal{U}$**-feasible t-matching:**

$$|F[U]| \leq \left\lfloor \frac{t|U|-1}{2} \right\rfloor \quad \forall U \in \mathcal{U}$$



Special Cases

- **Nonbipartite matching**
- **Triangle-free 2-matching** with edge multiplicity
- **Even factor**
- **Square-free 2-matching**
- **$K_{t,t}$-free t-matching**
- 2-matchings covering edge cuts
- Hamilton cycles

Solved:

- G: Bipartite
- $\forall U \in \mathcal{U}$ is "factor-critical"
- w is vertex-induced on $\forall U \in \mathcal{U}$

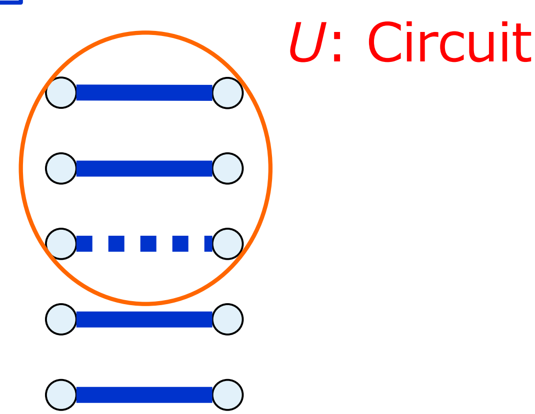
- ***Min-max theorem***
- ***LP with dual integrality***
- ***Combinatorial algorithm***

Further Research

- **Application to TSP**
  - ➢ Subtour elimination ≃ Blossom constraint
    ➔ **So what??**
  - ➢ New class of ***-free 2-factors??

    - C_4 [Hartvigsen '06, Pap '07]
    - C_6 with ≥2 chords [**T.** '16]

- **Matroids** as Special Cases
  - ➢ Matroids (t=1, $\mathcal{U}$={Circuit})
  - ➢ Arborescences
  - ➢ And more?? So what??



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