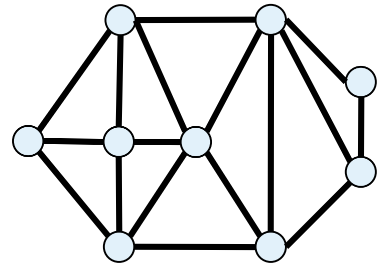
Finding a maximum 2-matching excluding prescribed cycles in bipartite graphs

Kenjiro Takazawa Hosei University

MFCS 2016 @ AGH University, Kraków
August 22-26, 2016

2-matching / 2-factor

$G = (V,E)$: Simple Undirected

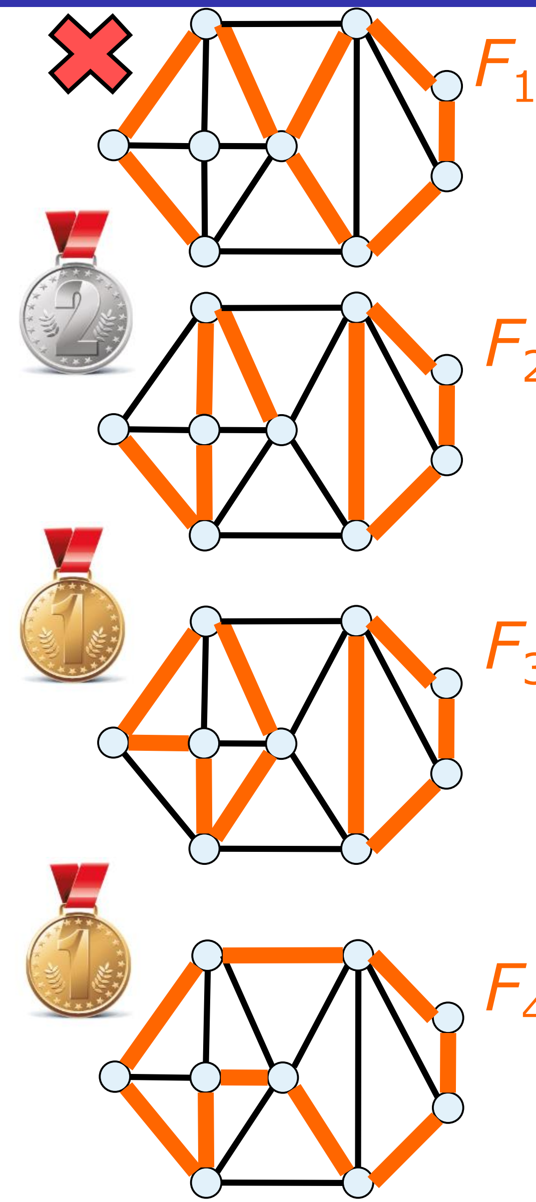


Def.

- $F \subseteq E$ is a **2-matching** $\Leftrightarrow \deg_F(v) \leq 2 \quad {}^\forall v \in V$
- $F \subseteq E$ is a **2-factor** $\Leftrightarrow \deg_F(v) = 2 \quad {}^\forall v \in V$

Cycle + Path

Cycle



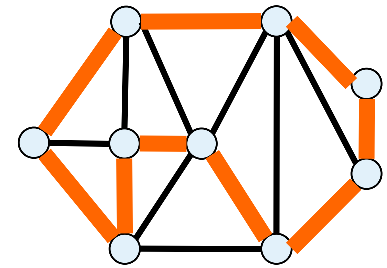
Problem

- Find a **2-factor**
- Find a **maximum 2-matching**

➢ **Solved in poly. time**
(Reduced to the matching problem)

Hamilton Cycle: Restricted 2-factor

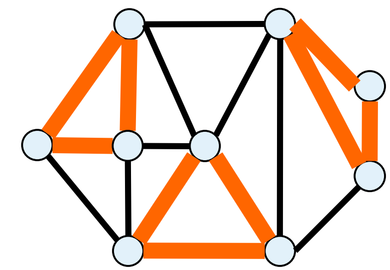
- 2-factor of **one cycle**



Hamilton cycle F^*

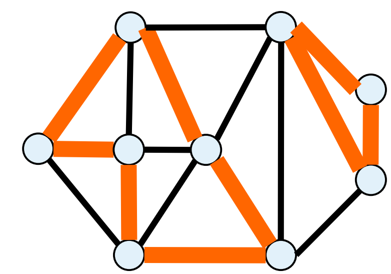
- 2-factor w/o $\boldsymbol{C_k}$ **for** $\boldsymbol{\forall k \leq n\text{-}1}$

- 2-factor w/o **subtour in** $\boldsymbol{\forall U \subsetneq V}$
    - **Subtour** in U = 2-factor in $G[U]$



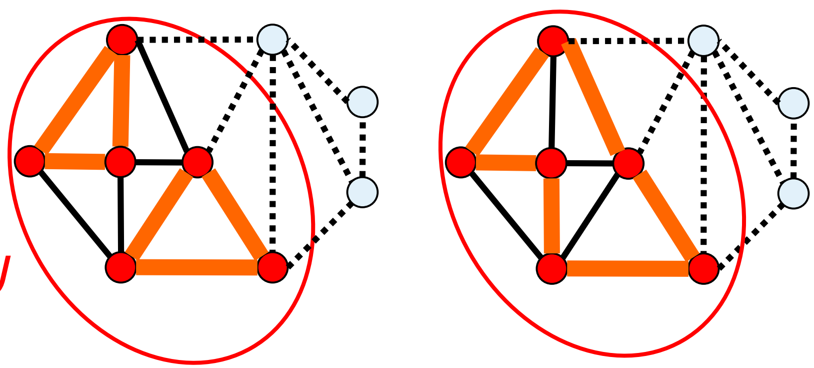
2-factor F_1

- 2-factor **covering** $^\forall$**edge cut**
    - F^* covers C
    - F_1, F_2 do not cover C

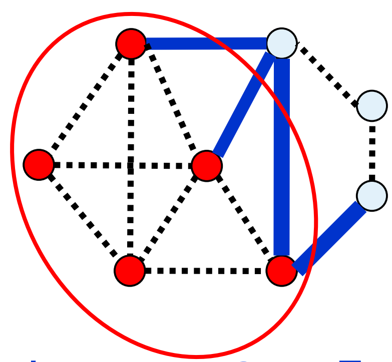


2-factor F_2

U

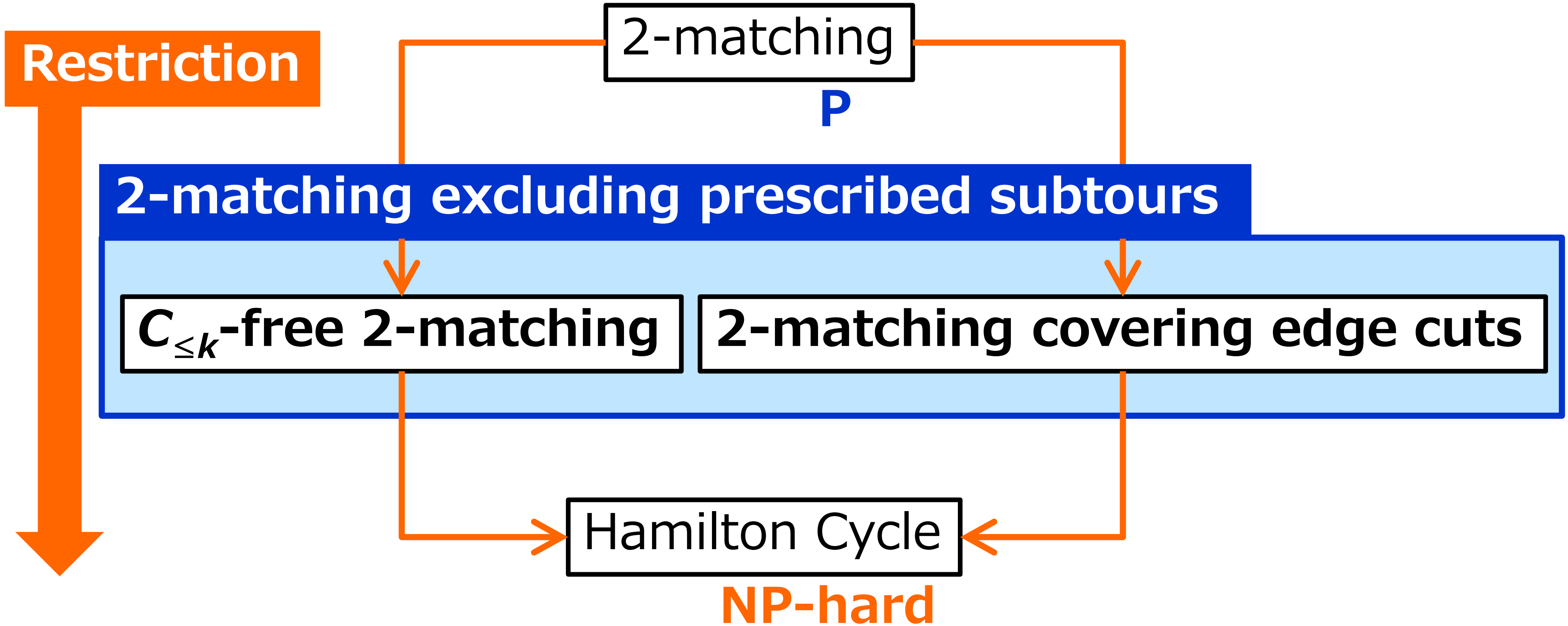


Subtours in U



Edge cut $C = \delta U$

Our Topic: Restricted 2-matchings



Our Goal

New framework of restricted 2-matchings

- **Solvable in poly. time**
- Extend the **matching theory**

Contents

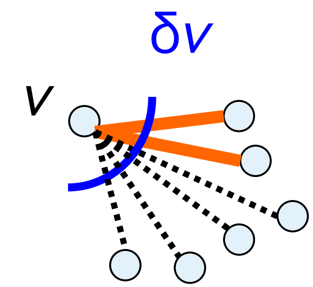
- **Introduction**: 2-matching and Hamilton cycle
- **Previous Work**
  - Subtour Elimination
  - $C_{\leq k}$-free 2-matching
  - $\mathcal{A}$-covering 2-factor
- **Our Framework**: $\mathcal{U}$-feasible 2-matching
  - Min-max Theorem
  - Combinatorial Algorithm
  - Decomposition Theorems
- **Conclusion**

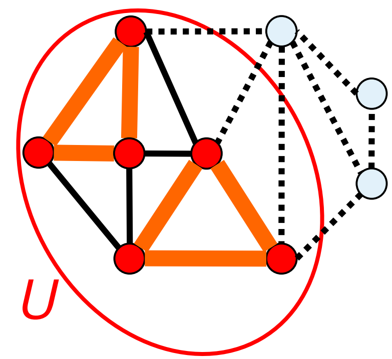
Subtour Elimination

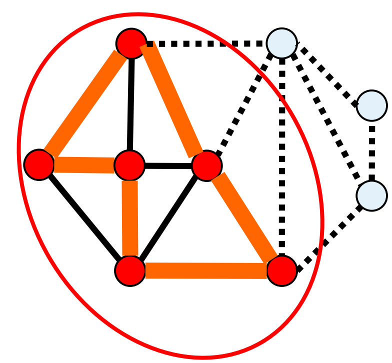
[Dantzig, Fulkerson, Johnson '54]
[Held, Karp '70]

Subtour Elimination Relaxation for TSP

min.	cx	
sub. to	$x(\delta v) = 2$	$v \in V$
	$\boldsymbol{x(E[U]) \leq \lvert U \rvert - 1}$	$\emptyset \subsetneq \boldsymbol{U} \subsetneq \boldsymbol{V}$
	$0 \leq x(e) \leq 1$	$e \in E$







- Standard LP relaxation for **TSP**
- **4/3-Conjecture**: The integrality gap is 4/3

[Goemans '95] etc.

$x \in \mathbf{R}^E$: $x(e) = \begin{cases} 1 & (e \in F) \\ 0 & (e \notin F) \end{cases}$

$x(E') = \sum_{e \in E'} x(e)$

$C_{\le k}$-free 2-matching

$k \in \mathbb{Z}_+$: Given integer

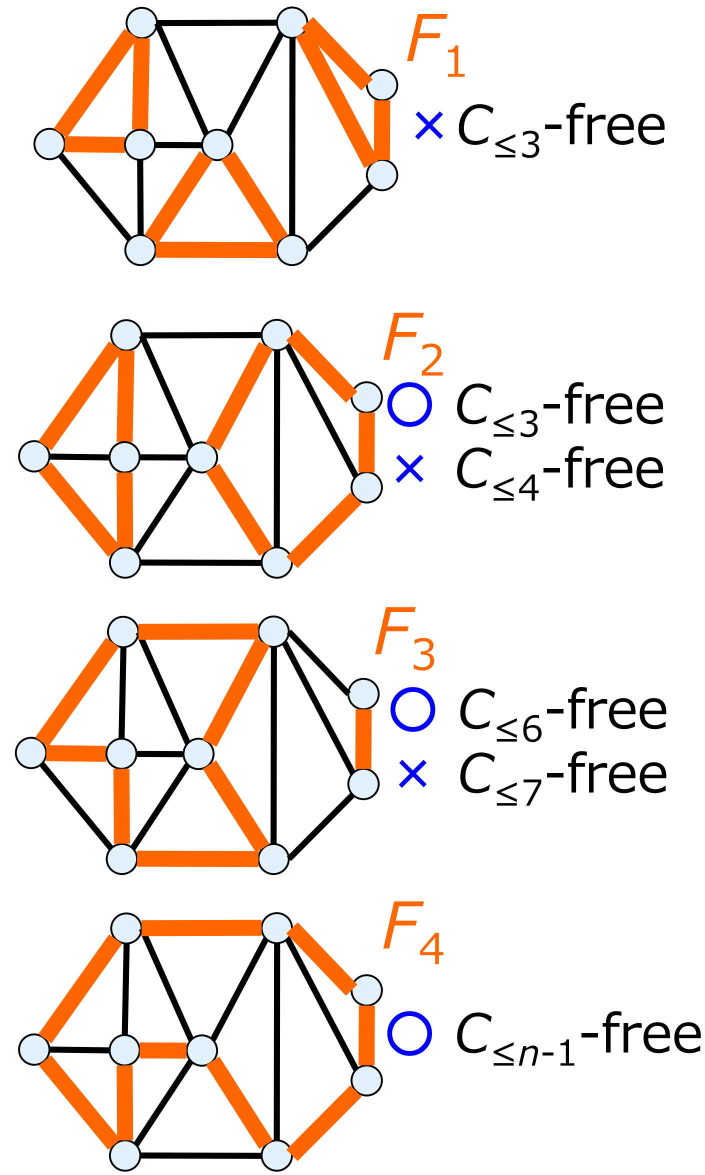
Def.

2-matching F is **$C_{\le k}$-free**
$\Leftrightarrow$ F excludes cycles of length $\le k$

Problem

- Find a **$C_{\le k}$-free 2-factor**
- Find a **max. $C_{\le k}$-free 2-matching**

➢ $k = 2$ ➔ **2-matching problem**
➢ $k = n\text{-}1$ ➔ **Hamilton cycle problem**



Application

Approximation algorithms for
- Graph-TSP
- Min. 2-edge connected subgraph

Complexity: Max $C_{\leq k}$-free 2-matching

	General Graph	Bipartite Graph
$k \geq n/2$	**NP-hard**	**NP-hard**
$k \geq 6$	**NP-hard***1	**NP-hard***3
$k = 5$	**NP-hard***1	---
$k = 4$	**Open**	**P***4
$k = 3$	**P***2	---
$k = 2$	**P**	**P**

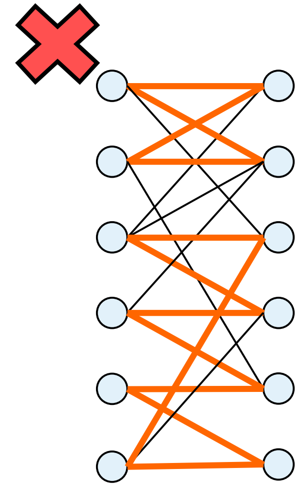
*1: [Papadimitriou '78]
*2: [Hartvigsen '84]
*3: [Geelen '99]
*4: [Hartvigsen '06], [Pap '07]

$C_{\leq 4}$-free 2-matching in bipartite graphs

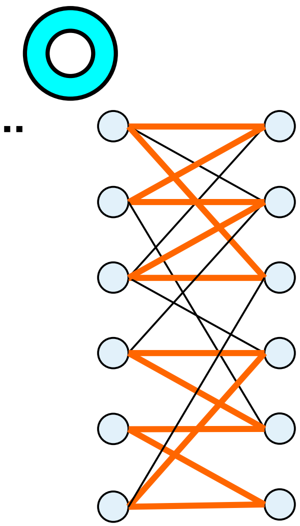
- **Well-solved**
- **Rich structure**

Theory of $C_{\leq 4}$-free 2-matching

- ***C*$_{\leq 4}$-free** 2-matchings in **Bipartite Graphs**:
  Classical Matching Theory is Extended



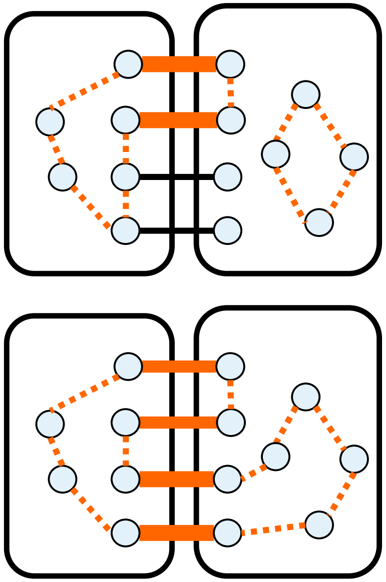
  - **Min-max theorem** [Király '99][Frank '03]
    - Kőnig, Tutte-Berge
  - **Combinatorial algorithm** [Hartvigsen '06][Pap '07]
    - Edmonds
  - **Decomposition theorem** [**T.** '15]
    - Dulmage-Mendelsohn, Edmonds-Gallai



◆ Max. Weight $C_{\leq 4}$-free 2-matching

- **NP-hard** in bipartite graphs
- Positive results for a certain class
  - **LP-formulation** w/ **dual integrality** [Makai '07]
    - Cunningham-Marsh
  - **Combinatorial algorithm** [**T.** '08]
  - **Discrete convexity** [Kobayashi, Szabó, **T.**: '12]

$\mathcal{A}$-covering 2-factor



Def. $\mathcal{A} \subseteq \mathbb{Z}$

2-factor F is **$\mathcal{A}$-covering**
$\overset{\text{def}}{\Leftrightarrow}$ F intersects every k-edge cut $^{\forall}k \in \mathcal{A}$

- **Hamilton cycle** = $\mathbb{Z}$-covering 2-factor

- 2-edge connected cubic graph:
  - **{3,4}-covering 2-factor** exists [Kaiser, Škrekovski '08], and can be found in $O(n^3)$ time [Boyd, Iwata, **T.** '13]
  - Min-weight **{3}-covering 2-factor** in $O(n^3)$ time [BI**T.** '13]

- Graphs w/o **{4,5}-covering 2-factor** [Čada, Chiba, Ozeki, Vrána, Yoshimoto '13]

➢ **Application**: Approximation of min. 2-edge connected subgraph [BI**T.** '13]

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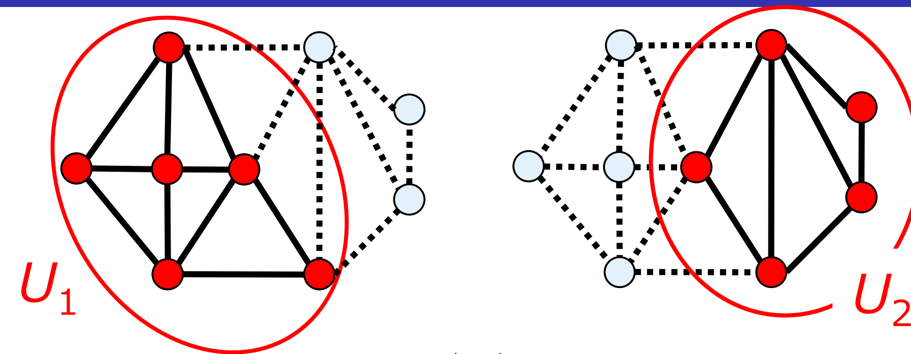
Our Framework: $\mathcal{U}$-feasible 2-matching

$\mathcal{U} = \{U_1, U_2, \ldots\} \subseteq 2^V$



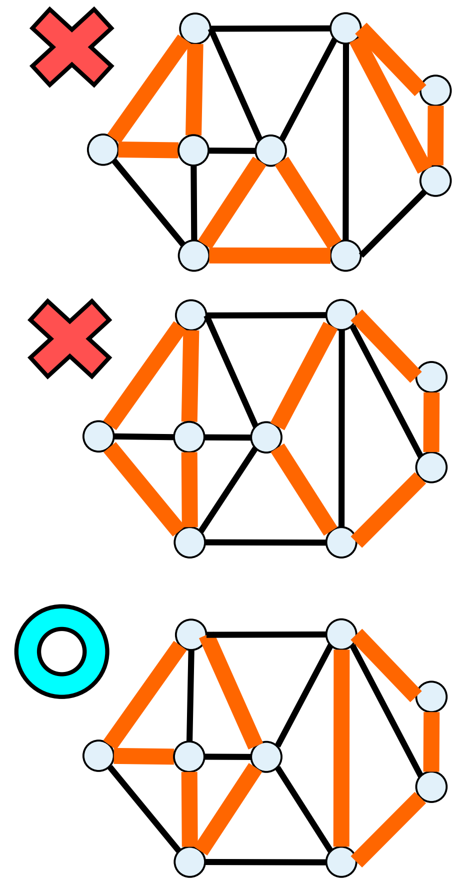
Def.

2-matching F is **$\mathcal{U}$-feasible**
$\Leftrightarrow$ $|F[U]| \leq |U| - 1 \ \forall U \in \mathcal{U}$

Subtour Elimination for $U \in \mathcal{U}$

Problem

- Find a **$\mathcal{U}$-feasible 2-factor**
- Find a **max. $\mathcal{U}$-feasible 2-matching**



- **Hamilton cycle**: $\mathcal{U} = 2^V \setminus \{V\}$
- **$C_{\leq k}$-free 2-matching**: $\mathcal{U} = \{U \subseteq V : |U| \leq k\}$
- **$\mathcal{A}$-covering 2-factor**: $\mathcal{U} = \{U \subseteq V : |\delta U| \in \mathcal{A}\}$

Our Goal: Poly. Solvable Class

Question

When is the problem solved in poly. time ?

- **Known Facts**:
  - $\mathcal{U} = 2^V \setminus \{V\}$: **NP-hard** [Hamilton cycle]
  - $\mathcal{U}=\{U\colon |U|\leq 4\}$, Bipartite: **P** [$C_{\leq 4}$-free 2-matching]
  - $\mathcal{U}=\{U\colon |\delta U| \in \{3,4\}\}$, 2-EC Cubic: **P** [{3,4}-covering 2-factor]

- **Our Answer**: **Extend the $C_{\leq 4}$-free case**
  - $\mathcal{U}=\{U\colon G[U]$ is **Hamilton-laceable**$\}$, Bipartite

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◆ **Polynomial in *n*, *m*, γ**
  - γ: Time for determining $\mathcal{U}$-feasibility
  - $|\mathcal{U}|$ might be exponential
  - $\mathcal{U}$ is given implicitly

Hamilton-Laceable Graphs

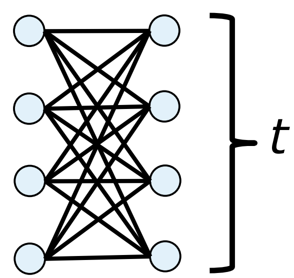
Def.

[Simmons '78]

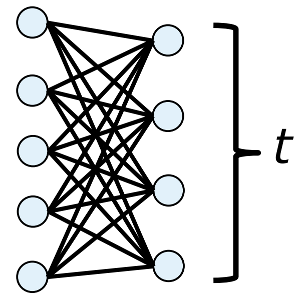
Bipartite graph $G=(V^+,V^-;E)$ is a **Hamilton-Laceable** $\Leftrightarrow$

- $|V^+|=|V^-|$, $^\forall u\in V^+$, $v\in V^-$, $^\exists$Hamilton path from u to v
- $|V^+|=|V^-|+1$, $^\forall u,v\in V^+$, $^\exists$Hamilton path from u to v

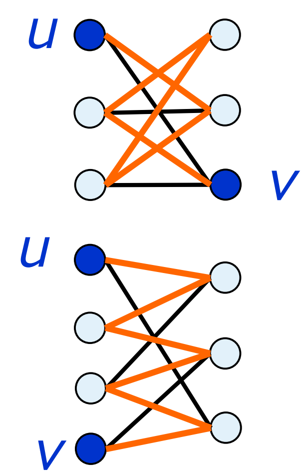
- $\boldsymbol{K_{t,t}}$ or $\boldsymbol{K_{t+1,t}}$ with $\leq \boldsymbol{t}$**-2** **edges** deleted [Simmons '78]
  - $\boldsymbol{C_4}$
  - $\boldsymbol{C_6}$ with ≥2 chords

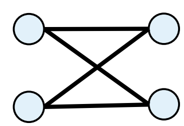


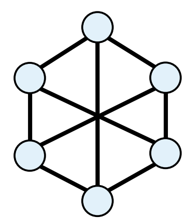
$K_{t,t}$

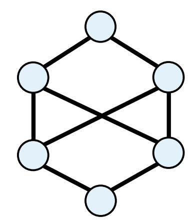


$K_{t+1,t}$

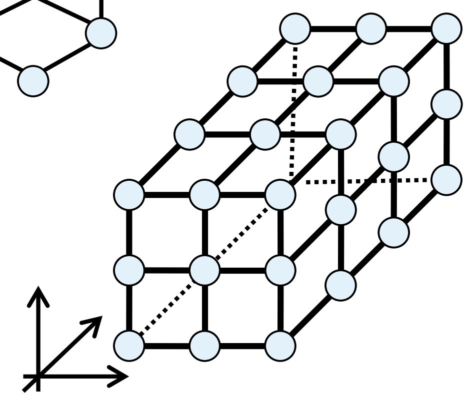








- ***d*-dimensional rectangular lattice**
  - Except for 2 x r, 3 x 2r (d=2) [Simmons '81]
  - O(dn) edges



Our Results

Assumption

G: **Bipartite**

$G[U]$ is **Hamilton-laceable** $\forall U \in \mathcal{U}$

Results

- **Min-max Theorem**
- **Combinatorial Algorithm**
- **Decomposition Theorems**

[Király '99]

[Hartvigsen '06][Pap '07]

[**T.** '15]

Extension of $C_{\leq 4}$-free 2-matching Theory

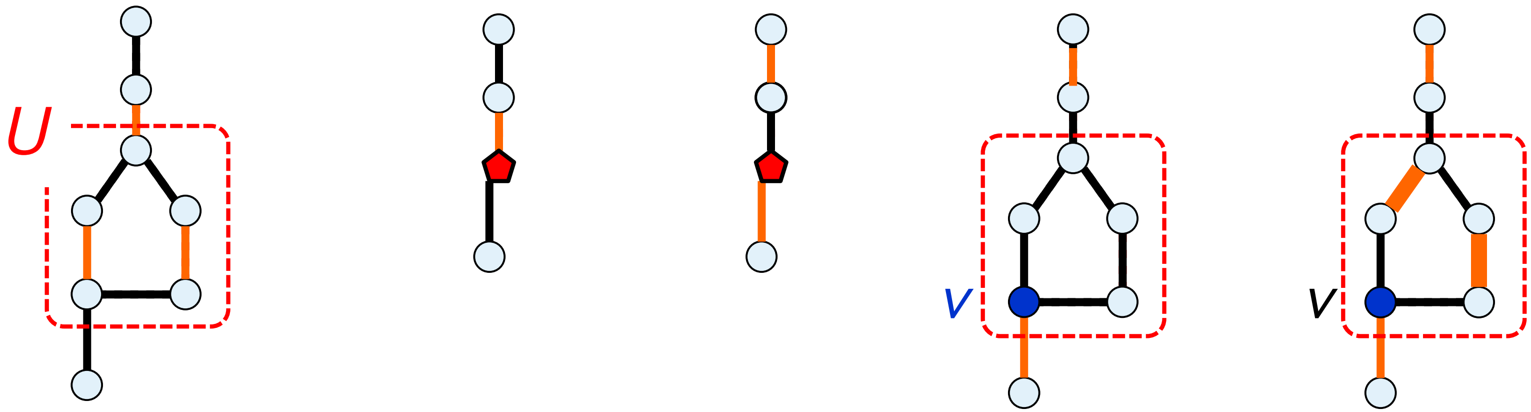
Subtour Elimination for $U \in \mathcal{U}$

Def. 2-matching F is a **$\mathcal{U}$-feasible** $\Leftrightarrow$ $|F[U]| \leq |U| - 1 \ \forall U \in \mathcal{U}$

Problem Find a **max. $\mathcal{U}$-feasible 2-matching**

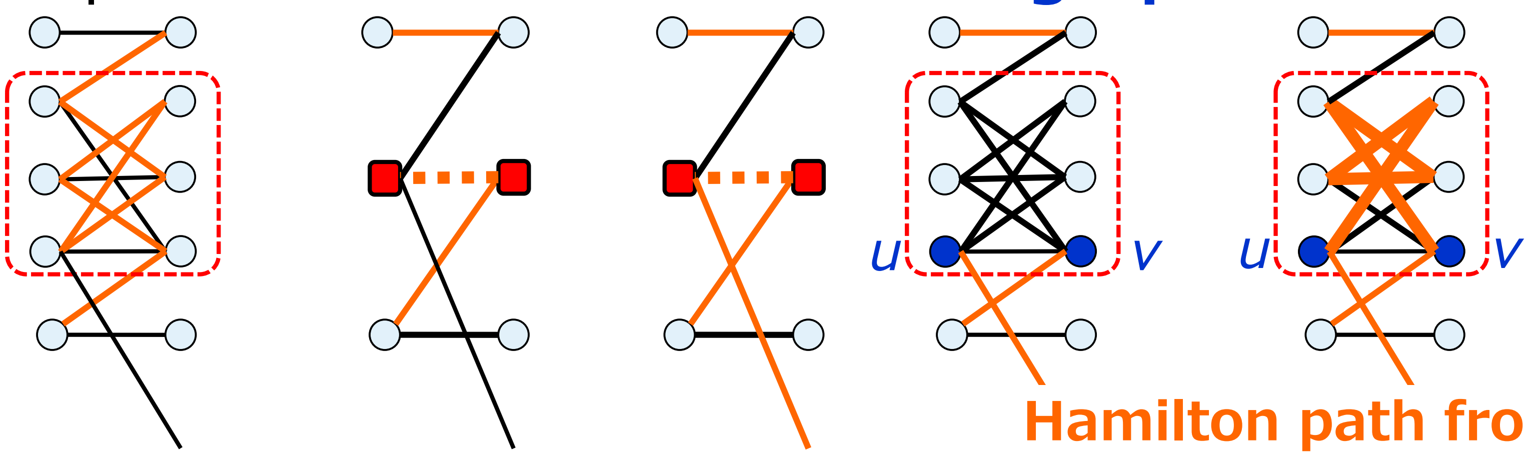
Algorithm Sketch

- **Why Hamilton-laceable graphs?**
  ➔ **Shrinking technique** works!
  - Nonbipartite matching : **Factor-critical graph** [Edmonds '65]



Matching covering *U*-*v*

  - Our problem: **Hamilton-laceable graph**



Hamilton path from *u* to *v*

Algorithmic Results

G: Bipartite

Theorem

If $G[U]$ is Hamilton-laceable $\forall U \in \mathcal{U}$,
max. **$\mathcal{U}$-feasible** 2-matching is found in **$O(n^3\gamma + n^2m)$ time**

If $G[C]$ is Hamilton-laceable $\forall$cycle C of length $\leq k$,
max. **$C_{\leq k}$-free** 2-matching is found in **$O(kn^3 + n^2m)$ time**

Corollary

Max. 2-matching excluding C_4 and C_6 **with ≥2 chords**
is found in $O(n^2m)$ time

In every ***d*-regular bipartite graph** with $d \geq 4$,
a **2-factor** excluding C_4 and C_6 **with ≥2 chords** exists and
can be found in $O(n^2m)$ time

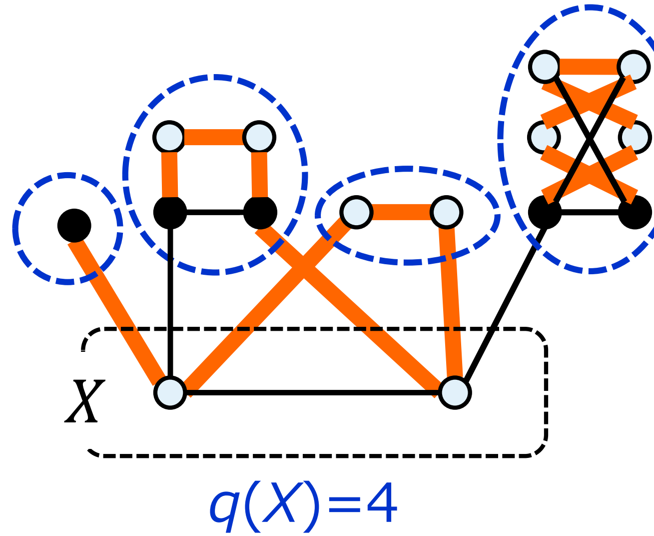
- **First positive results for $C_{\leq 6}$-free 2-matching**

Min-Max Theorem

Theorem

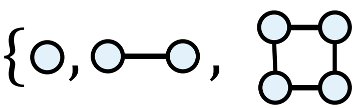
max $\{|F| : F$ is a $\mathcal{U}$-feasible 2-matching$\}$
$= \min\{|V|+|X| - q(X) : X \subseteq V\}$

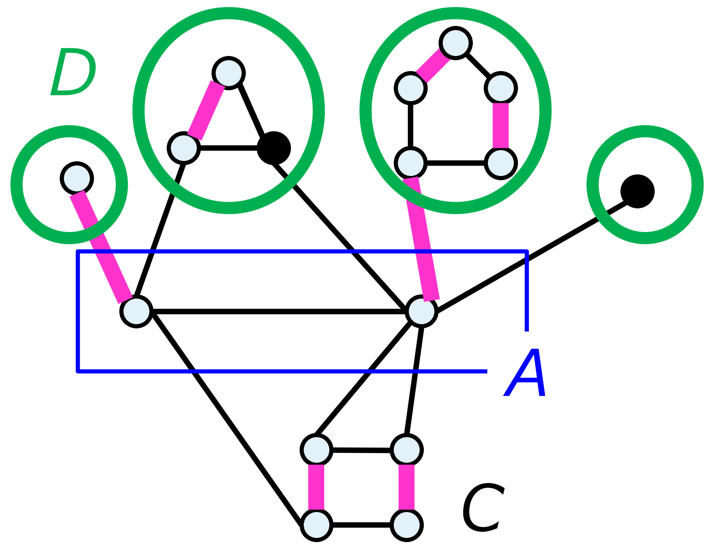
$q(X) = \#\{\text{Component } C \text{ in } G[\bar{X}] : V(C) \in \mathcal{U}\}$



Theorem [Király '99]

max $\{|M| : M$ is a C_4-free 2-matching$\}$
$= \min\{|V|+ |X| - q_4(X) : X \subseteq V\}$

$q_4(X) = \#\{$  in $G[\bar{X}]\}$



Theorem [Tutte '47, Berge '58]

$\max\{|M| : M \text{ is a matching}\}$
$= \frac{1}{2}\min\{|V| + |X| - \mathrm{odd}(X) : X \subseteq V\}$

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Conclusion

$\mathcal{U}$-feasible 2-matching

- **New framework** of restricted 2-matching
- Excludes **subtours** for $^\forall U \in \mathcal{U} \subset 2^V$
- Classical matching theory is extended if
  - G is bipartite and $G[U]$ is Hamilton-laceable $^\forall U \in \mathcal{U}$

Future Work

- **Weighted** version
  - LP, Algorithm, Discrete Convexity
- **Broader** solvable class than Hamilton-laceable graphs
- **Application**
  - Approximation for TSP etc.

END of Slides