

Finding a maximum 2-matching excluding prescribed cycles in bipartite graphs

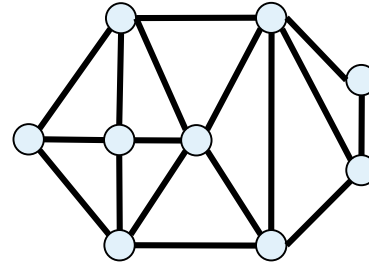
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2-matching / 2-factor

$G = (V, E)$: Simple Undirected



Def.

● $F \subseteq E$ is a **2-matching**

$$\Leftrightarrow \deg_F(v) \leq 2 \quad \forall v \in V$$

● $F \subseteq E$ is a **2-factor**

$$\Leftrightarrow \deg_F(v) = 2 \quad \forall v \in V$$

Cycle + Path

Cycle

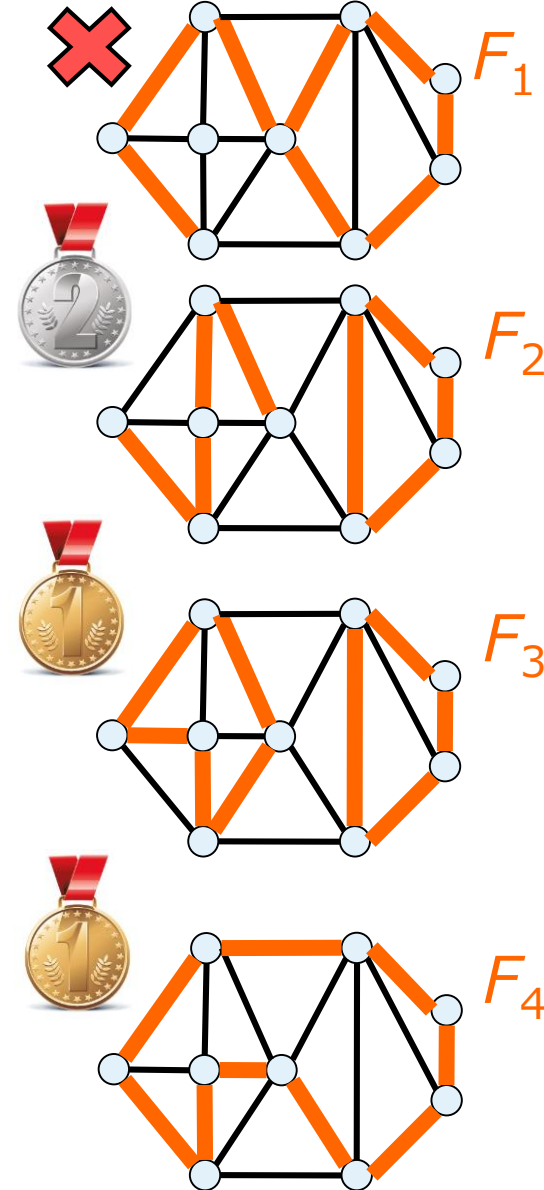
Problem

● Find a **2-factor**

● Find a **maximum 2-matching**

➤ **Solved in poly. time**

(Reduced to the matching problem)



Hamilton Cycle: Restricted 2-factor

➤ 2-factor of **one cycle**

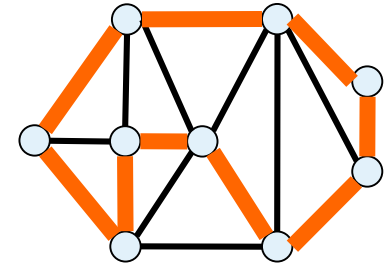
➤ 2-factor w/o C_k for $\forall k \leq n-1$

➤ 2-factor w/o **subtour** in $\forall U \subsetneq V$

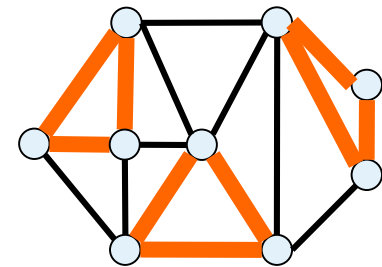
- **Subtour** in $U = 2$ -factor in $G[U]$

➤ 2-factor **covering** \forall **edge cut**

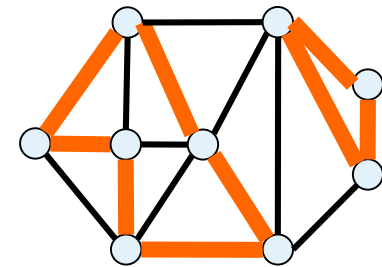
- F^* covers C
- F_1, F_2 do not cover C



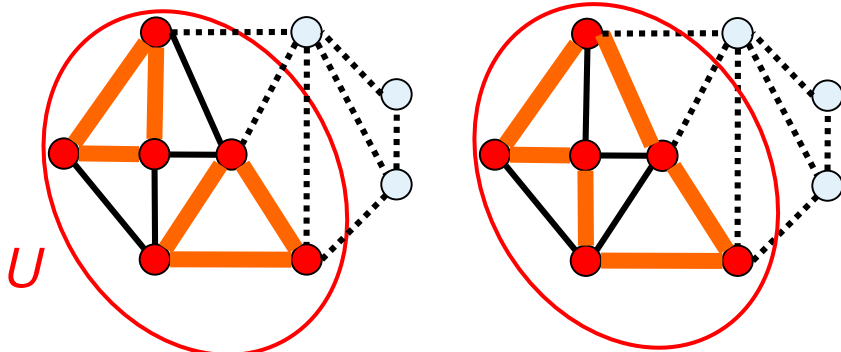
Hamilton cycle F^*



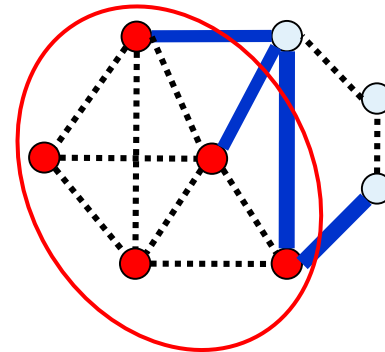
2-factor F_1



2-factor F_2



Subtours in U



Edge cut $C = \delta U$

Restriction

2-matching

P

2-matching excluding prescribed subtours

$C_{\leq k}$ -free 2-matching

2-matching covering edge cuts

Hamilton Cycle

NP-hard

Our Goal

New framework of restricted 2-matchings

- **Solvable in poly. time**
- Extend the **matching theory**

- **Introduction:** 2-matching and Hamilton cycle
- **Previous Work**
 - Subtour Elimination
 - $C_{\leq k}$ -free 2-matching
 - A -covering 2-factor
- **Our Framework:** \mathcal{U} -feasible 2-matching
 - Min-max Theorem
 - Combinatorial Algorithm
 - Decomposition Theorems
- **Conclusion**

[Dantzig, Fulkerson, Johnson '54]

[Held, Karp '70]

Subtour Elimination Relaxation for TSP

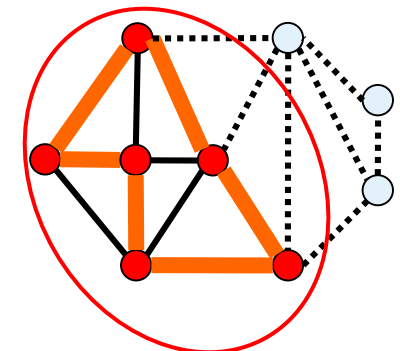
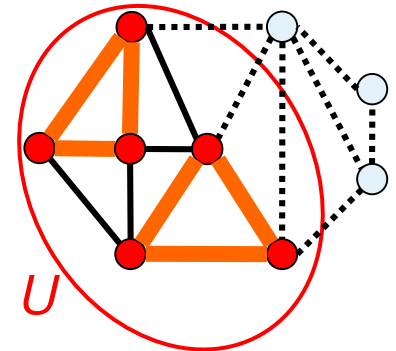
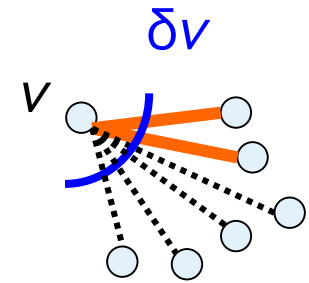
$$\text{min. } cx$$

$$\text{sub. to } x(\delta v) = 2 \quad v \in V$$

$$x(E[U]) \leq |U| - 1 \quad \emptyset \subsetneq U \subsetneq V$$

$$0 \leq x(e) \leq 1 \quad e \in E$$

- Standard LP relaxation for **TSP**
- **4/3-Conjecture**: The integrality gap is 4/3
[Goemans '95] etc.



$$x \in \mathbf{R}^E: x(e) = \begin{cases} 1 & (e \in F) \\ 0 & (e \notin F) \end{cases}$$

$$x(E') = \sum_{e \in E'} x(e)$$

$C_{\leq k}$ -free 2-matching

$k \in \mathbb{Z}_+$: Given integer

Def.

2-matching F is **$C_{\leq k}$ -free**

$\Leftrightarrow F$ excludes cycles of length $\leq k$

Problem

● Find a **$C_{\leq k}$ -free 2-factor**

● Find a **max. $C_{\leq k}$ -free 2-matching**

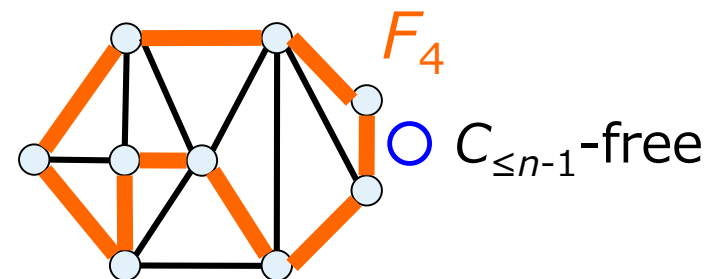
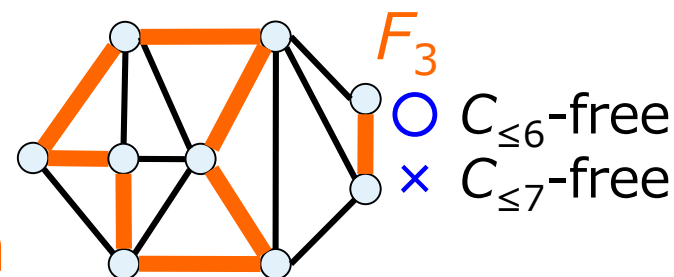
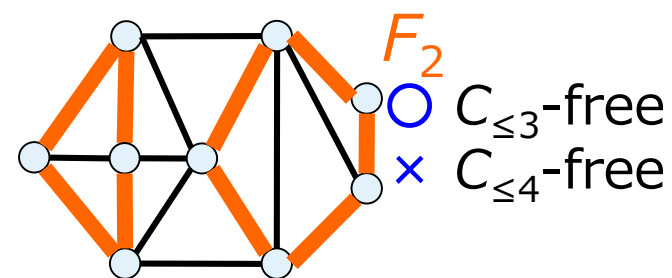
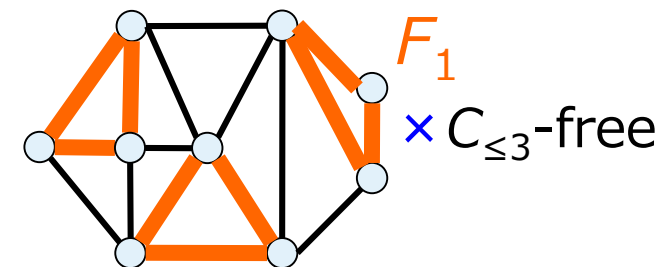
➤ $k = 2 \rightarrow$ **2-matching problem**

➤ $k = n-1 \rightarrow$ **Hamilton cycle problem**

Application

Approximation algorithms for

- Graph-TSP
- Min. 2-edge connected subgraph



Complexity: Max $C_{\leq k}$ -free 2-matching

	General Graph	Bipartite Graph
$k \geq n/2$	NP-hard	NP-hard
$k \geq 6$	NP-hard *1	NP-hard *3
$k = 5$	NP-hard *1	---
$k = 4$	Open	P *4
$k = 3$	P *2	---
$k = 2$	P	P

*1: [Papadimitriou '78]

*2: [Hartvigsen '84]

*3: [Geelen '99]

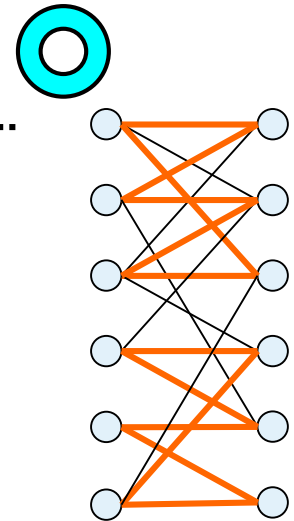
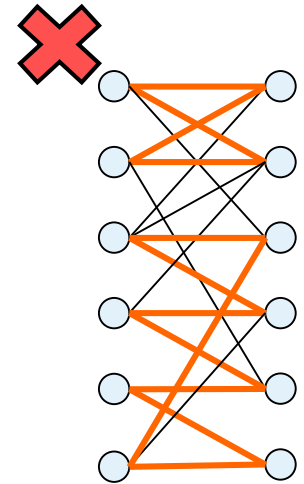
*4: [Hartvigsen '06], [Pap '07]

**$C_{\leq 4}$ -free 2-matching
in bipartite graphs**

- **Well-solved**
- **Rich structure**

- $C_{\leq 4}$ -free 2-matchings in **Bipartite Graphs**:
Classical Matching Theory is Extended

- **Min-max theorem** [Király '99][Frank '03]
 - König, Tutte-Berge
- **Combinatorial algorithm** [Hartvigsen '06][Pap '07]
 - Edmonds
- **Decomposition theorem** [T. '15]
 - Dulmage-Mendelsohn, Edmonds-Gallai



- ◆ **Max. Weight $C_{\leq 4}$ -free 2-matching**

- **NP-hard** in bipartite graphs
- Positive results for a certain class
 - **LP-formulation w/ dual integrality** [Makai '07]
 - Cunningham-Marsh
 - **Combinatorial algorithm** [T. '08]
 - **Discrete convexity** [Kobayashi, Szabó, T.: '12]

Def. $A \subseteq \mathbb{Z}$

2-factor F is **A-covering**

def
 $\Leftrightarrow F$ intersects every k -edge cut $\forall k \in A$

● **Hamilton cycle** = \mathbb{Z} -covering 2-factor

● 2-edge connected cubic graph:

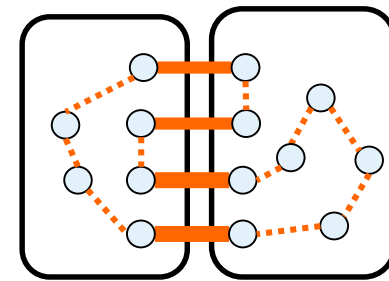
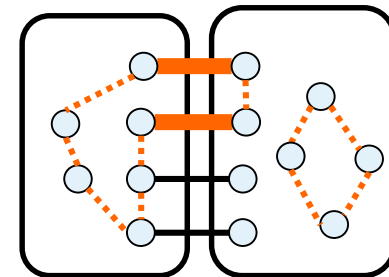
- **{3,4}-covering 2-factor** exists [Kaiser, Škrekovski '08],
and can be found in $O(n^3)$ time [Boyd, Iwata, T. '13]
- Min-weight **{3}-covering 2-factor** in $O(n^3)$ time [BIT. '13]

● Graphs w/o **{4,5}-covering 2-factor**

[Čada, Chiba, Ozeki, Vrána, Yoshimoto '13]

➤ **Application:** Approximation of min. 2-edge connected subgraph

[BIT. '13]



- **Introduction:** 2-matching and Hamilton cycle
- **Previous Work**
 - Subtour Elimination
 - $C_{\leq k}$ -free 2-matching
 - A -covering 2-factor
- **Our Framework:** \mathcal{U} -feasible 2-matching
 - Min-max Theorem
 - Combinatorial Algorithm
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- **Conclusion**

Our Framework: \mathcal{U} -feasible 2-matching

$$\mathcal{U} = \{U_1, U_2, \dots\} \subseteq 2^V$$

Def.

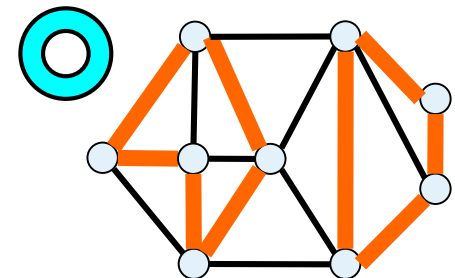
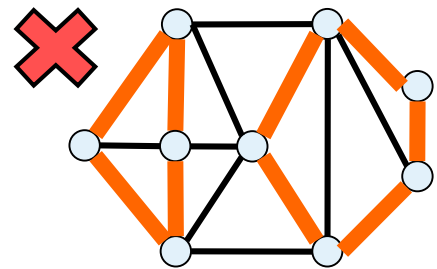
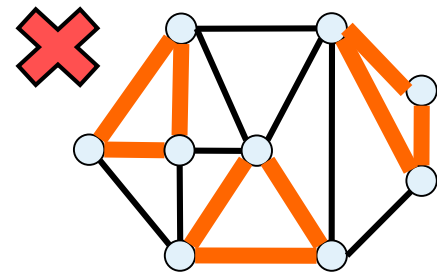
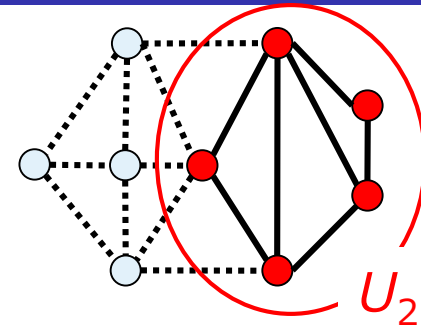
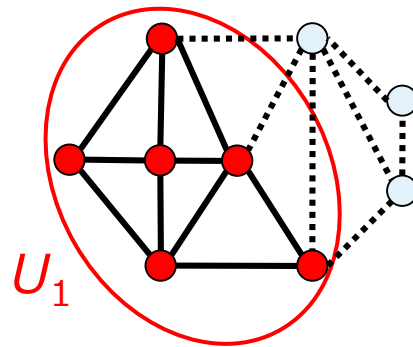
2-matching F is **\mathcal{U} -feasible**
 $\Leftrightarrow |F[U]| \leq |U| - 1 \quad \forall U \in \mathcal{U}$

Subtour Elimination for $U \in \mathcal{U}$

Problem

- Find a **\mathcal{U} -feasible 2-factor**
- Find a **max. \mathcal{U} -feasible 2-matching**

- **Hamilton cycle:** $\mathcal{U} = 2^V \setminus \{V\}$
- **$C_{\leq k}$ -free 2-matching:** $\mathcal{U} = \{U \subseteq V : |U| \leq k\}$
- **\mathcal{A} -covering 2-factor:** $\mathcal{U} = \{U \subseteq V : |\delta U| \in \mathcal{A}\}$



Question

When is the problem solved in poly. time ?

● Known Facts:

- $\mathcal{U} = 2^V \setminus \{V\}$: **NP-hard** [Hamilton cycle]
- $\mathcal{U} = \{U: |U| \leq 4\}$, Bipartite: **P** [$C_{\leq 4}$ -free 2-matching]
- $\mathcal{U} = \{U: |\delta U| \in \{3,4\}\}$, 2-EC Cubic: **P** [$\{3,4\}$ -covering 2-factor]

● Our Answer: **Extend the $C_{\leq 4}$ -free case**

- $\mathcal{U} = \{U: G[U] \text{ is Hamilton-laceable}\}$, Bipartite

◆ Polynomial in n, m, γ

- γ : Time for determining \mathcal{U} -feasibility
- $|\mathcal{U}|$ might be exponential
- \mathcal{U} is given implicitly

Def.

[Simmons '78]

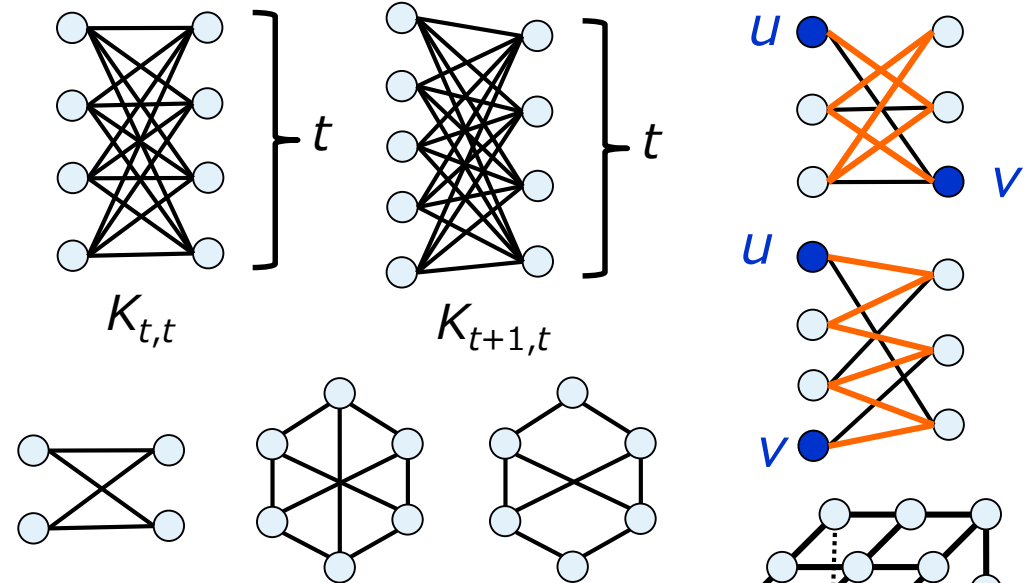
Bipartite graph $G=(V^+,V^-;E)$ is a **Hamilton-Laceable** \Leftrightarrow

- $|V^+|=|V^-|$, $\forall u \in V^+, v \in V^-$, \exists Hamilton path from u to v
- $|V^+|=|V^-|+1$, $\forall u, v \in V^+$, \exists Hamilton path from u to v

- $K_{t,t}$ or $K_{t+1,t}$
with $\leq t-2$ edges deleted
[Simmons '78]

➤ C_4

➤ C_6 with ≥ 2 chords



- d -dimensional rectangular lattice

- Except for $2 \times r$, $3 \times 2r$ ($d=2$) [Simmons '81]
- $O(dn)$ edges

Assumption

G : **Bipartite**

$G[U]$ is **Hamilton-laceable** $\forall U \in \mathcal{U}$

Results

- **Min-max Theorem**
- **Combinatorial Algorithm**
- **Decomposition Theorems**

[Király '99]

[Hartvigsen '06][Pap '07]

[T. '15]



Extension of $C_{\leq 4}$ -free 2-matching Theory

Subtour Elimination for $U \in \mathcal{U}$

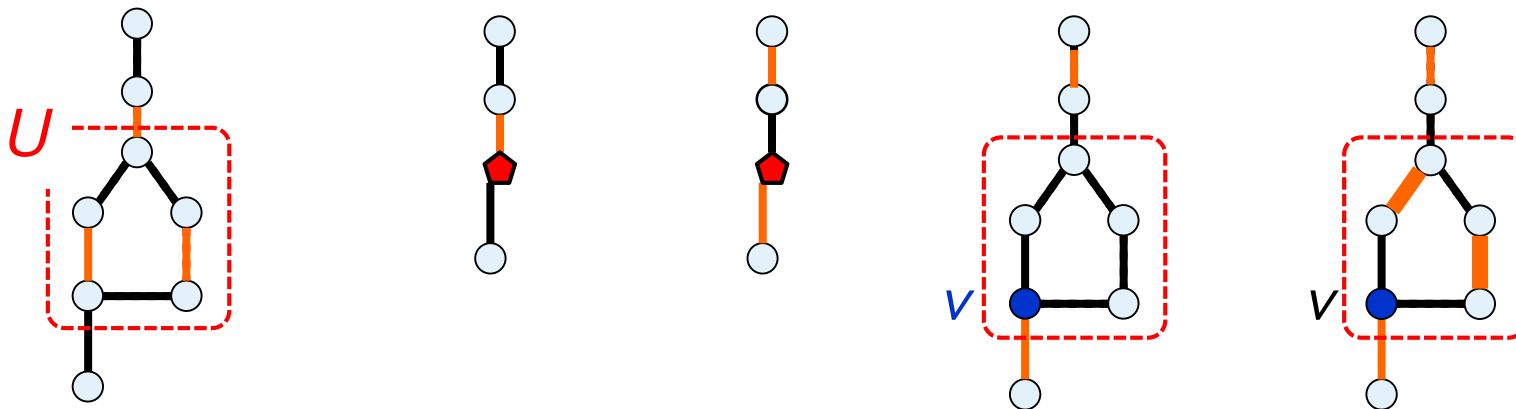
Def. 2-matching F is a **\mathcal{U} -feasible** $\Leftrightarrow |F[U]| \leq |U| - 1 \forall U \in \mathcal{U}$

Problem Find a **max. \mathcal{U} -feasible 2-matching**

- Why Hamilton-laceable graphs?

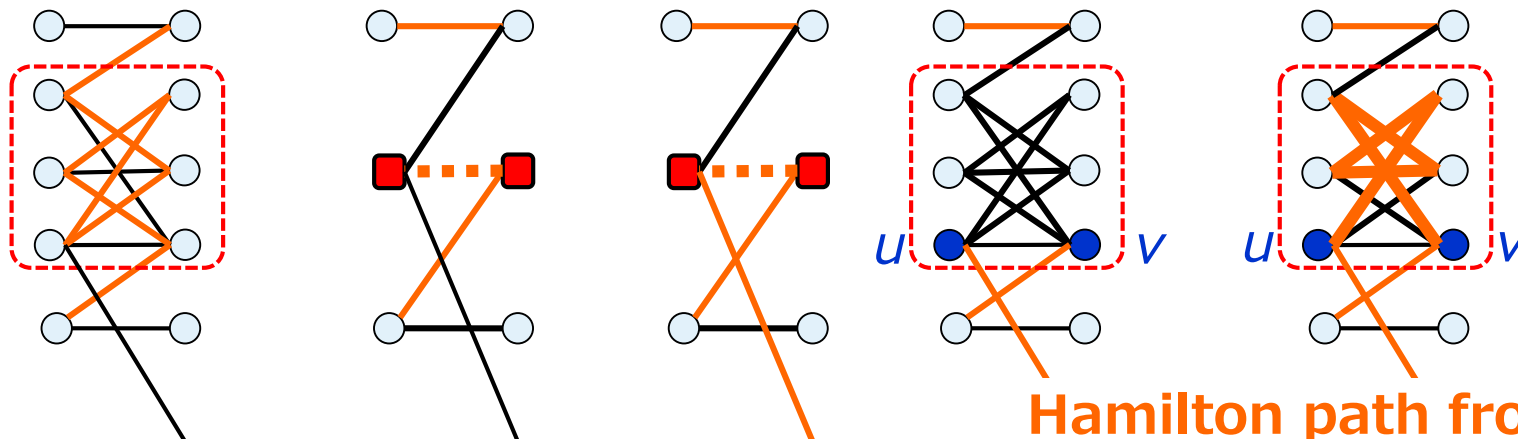
- Shrinking technique works!

- Nonbipartite matching : **Factor-critical graph** [Edmonds '65]



Matching covering $U-v$

- Our problem: **Hamilton-laceable graph**



Hamilton path from u to v

G : Bipartite

Theorem

If $G[U]$ is Hamilton-laceable $\forall U \in \mathcal{U}$,
max. **\mathcal{U} -feasible** 2-matching is found in **$O(n^3\gamma + n^2m)$ time**

If $G[C]$ is Hamilton-laceable \forall cycle C of length $\leq k$,
max. **$C_{\leq k}$ -free** 2-matching is found in **$O(kn^3 + n^2m)$ time**

Corollary

Max. 2-matching excluding C_4 and C_6 with ≥ 2 chords
is found in **$O(n^2m)$ time**

In every **d -regular bipartite graph** with $d \geq 4$,
a **2-factor** excluding C_4 and C_6 with ≥ 2 chords exists and
can be found in **$O(n^2m)$ time**

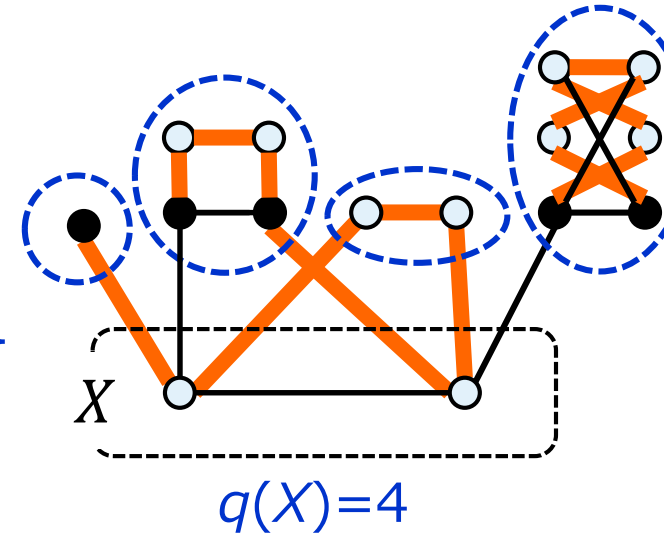
➤ **First positive results for $C_{\leq 6}$ -free 2-matching**

Theorem

$$\max \{ |F| : F \text{ is a } \mathcal{U}\text{-feasible 2-matching} \}$$

$$= \min \{ |V| + |X| - q(X) : X \subseteq V \}$$

$$q(X) = \#\{\text{Component } C \text{ in } G[\bar{X}] : V(C) \in \mathcal{U}\}$$



Theorem [Király '99]

$$\max \{ |M| : M \text{ is a } C_4\text{-free 2-matching} \}$$

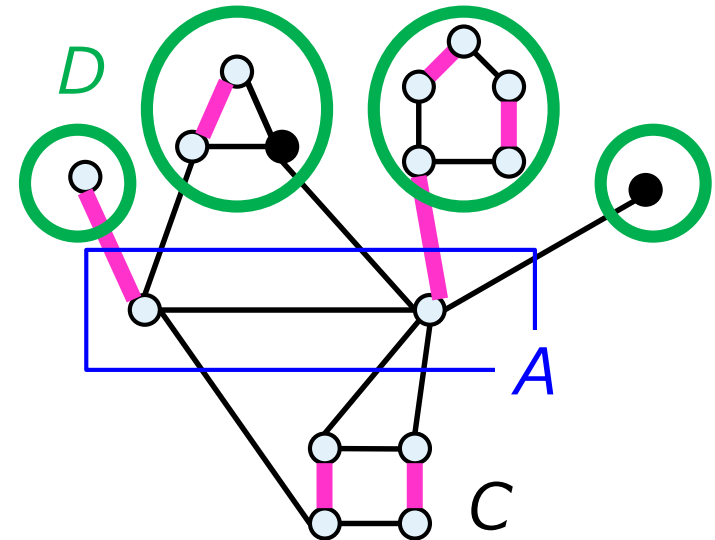
$$= \min \{ |V| + |X| - q_4(X) : X \subseteq V \}$$

$$q_4(X) = \#\{ \circ, \circ-\circ, \begin{array}{c} \circ \quad \circ \\ | \quad | \\ \circ \quad \circ \end{array} \text{ in } G[\bar{X}] \}$$

Theorem [Tutte '47, Berge '58]

$$\max \{ |M| : M \text{ is a matching} \}$$

$$= \frac{1}{2} \min \{ |V| + |X| - \text{odd}(X) : X \subseteq V \}$$



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- **Conclusion**

\mathcal{U} -feasible 2-matching

- **New framework** of restricted 2-matching
- Excludes **subtours** for $\forall U \in \mathcal{U} \subset 2^V$
- **Classical matching theory** is extended if
 - G is **bipartite** and $G[U]$ is **Hamilton-laceable** $\forall U \in \mathcal{U}$

Future Work

- **Weighted** version
 - LP, Algorithm, Discrete Convexity
- **Broader** solvable class than Hamilton-laceable graphs
- **Application**
 - Approximation for TSP etc.

