

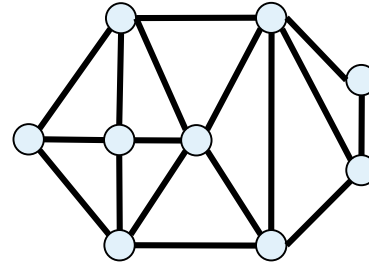
# Finding a maximum 2-matching excluding prescribed cycles in bipartite graphs

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# 2-matching / 2-factor

$G = (V, E)$ : Simple Undirected



## Def.

●  $F \subseteq E$  is a **2-matching**

$$\Leftrightarrow \deg_F(v) \leq 2 \quad \forall v \in V$$

●  $F \subseteq E$  is a **2-factor**

$$\Leftrightarrow \deg_F(v) = 2 \quad \forall v \in V$$

Cycle + Path

Cycle

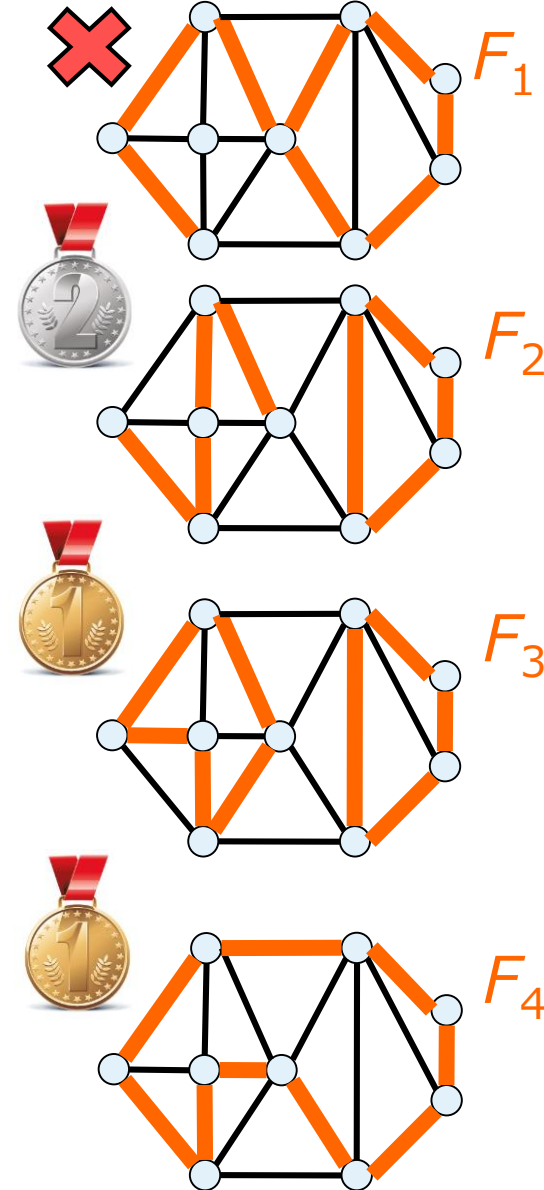
## Problem

● Find a **2-factor**

● Find a **maximum 2-matching**

➤ **Solved in poly. time**

(Reduced to the matching problem)



# Hamilton Cycle: Restricted 2-factor

➤ 2-factor of **one cycle**

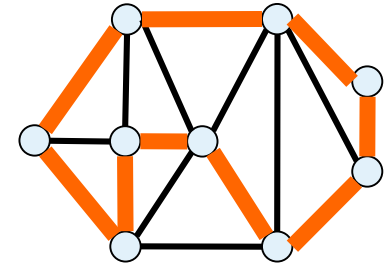
➤ 2-factor w/o  $C_k$  for  $\forall k \leq n-1$

➤ 2-factor w/o **subtour** in  $\forall U \subsetneq V$

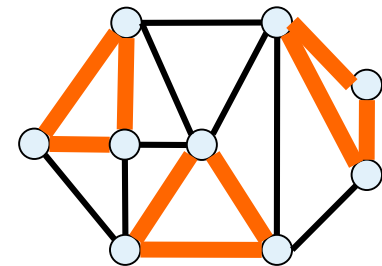
- **Subtour** in  $U = 2$ -factor in  $G[U]$

➤ 2-factor **covering**  $\forall$  **edge cut**

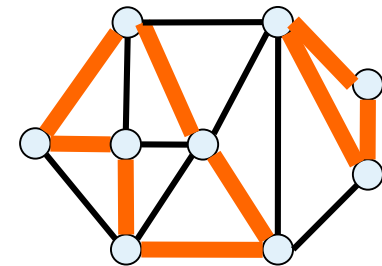
- $F^*$  covers  $C$
- $F_1, F_2$  do not cover  $C$



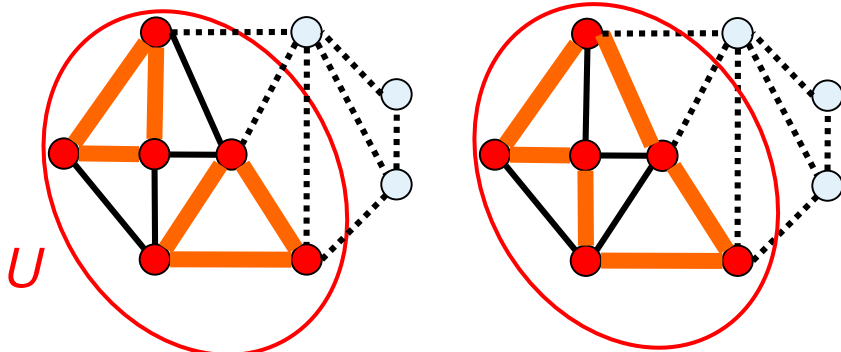
Hamilton cycle  $F^*$



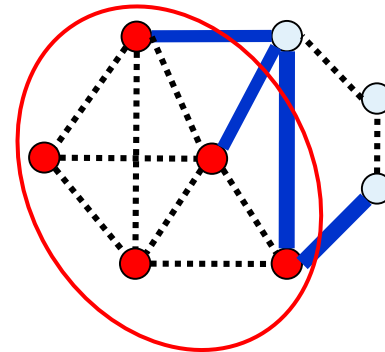
2-factor  $F_1$



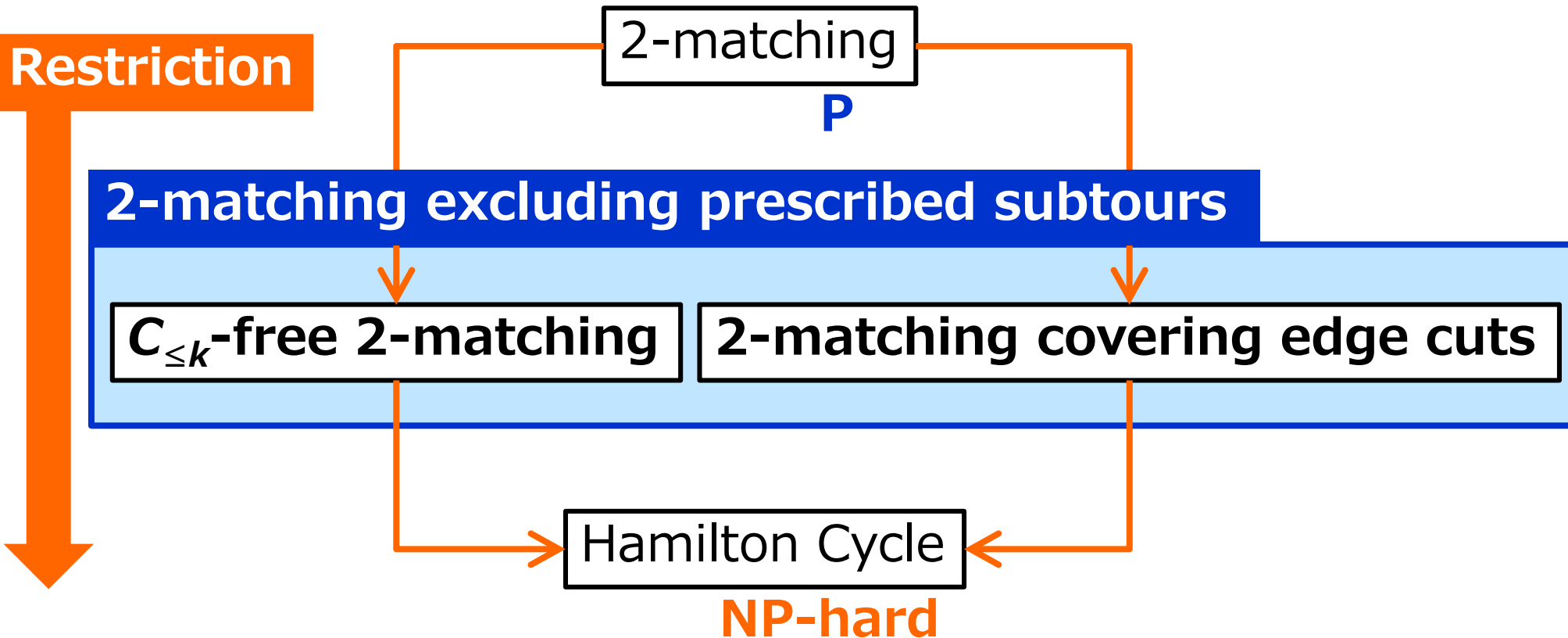
2-factor  $F_2$



Subtours in  $U$



Edge cut  $C = \delta U$



## Our Goal

**New framework** of restricted 2-matchings

- **Solvable in poly. time**
- Extend the **matching theory**

- **Introduction:** 2-matching and Hamilton cycle
- **Previous Work**
  - Subtour Elimination
  - $C_{\leq k}$ -free 2-matching
  - $A$ -covering 2-factor
- **Our Framework:**  $\mathcal{U}$ -feasible 2-matching
  - Min-max Theorem
  - Combinatorial Algorithm
  - Decomposition Theorems
- **Conclusion**

[Dantzig, Fulkerson, Johnson '54]

[Held, Karp '70]

## Subtour Elimination Relaxation for TSP

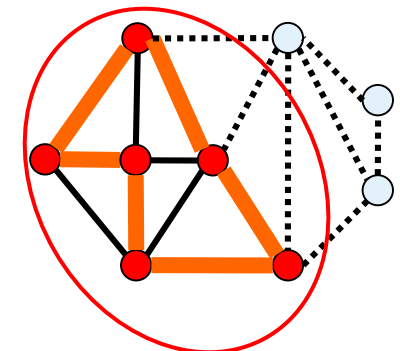
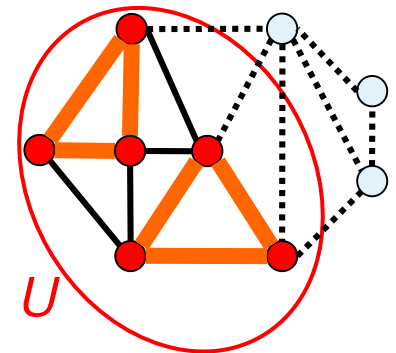
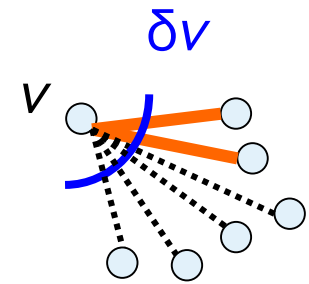
$$\text{min. } cx$$

$$\text{sub. to } x(\delta v) = 2 \quad v \in V$$

$$x(E[U]) \leq |U| - 1 \quad \emptyset \subsetneq U \subsetneq V$$

$$0 \leq x(e) \leq 1 \quad e \in E$$

- Standard LP relaxation for **TSP**
- **4/3-Conjecture**: The integrality gap is 4/3  
[Goemans '95] etc.



$$x \in \mathbf{R}^E: x(e) = \begin{cases} 1 & (e \in F) \\ 0 & (e \notin F) \end{cases}$$

$$x(E') = \sum_{e \in E'} x(e)$$

# $C_{\leq k}$ -free 2-matching

$k \in \mathbb{Z}_+$ : Given integer

## Def.

2-matching  $F$  is  **$C_{\leq k}$ -free**  
 $\Leftrightarrow F$  excludes cycles of length  $\leq k$

## Problem

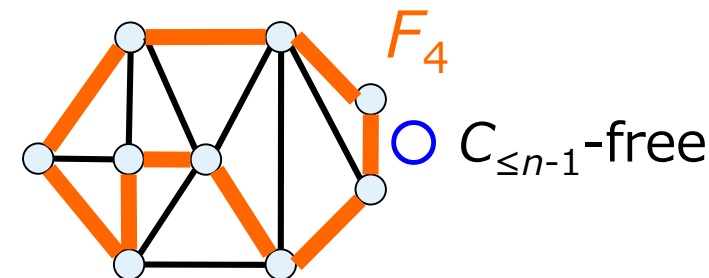
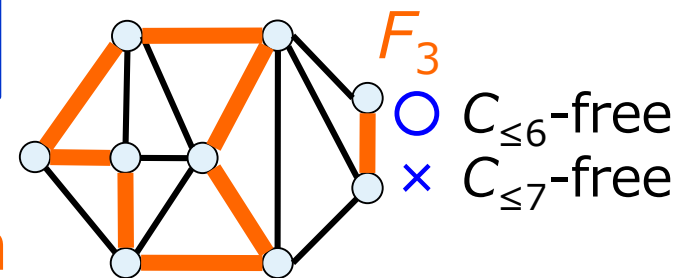
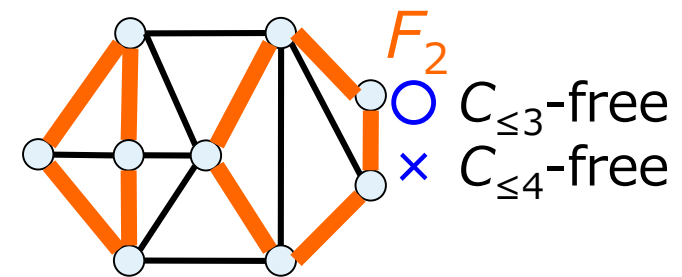
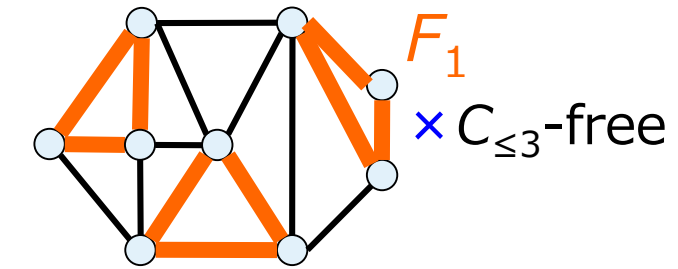
- Find a  **$C_{\leq k}$ -free 2-factor**
- Find a **max.  $C_{\leq k}$ -free 2-matching**

- $k = 2 \rightarrow$  **2-matching problem**
- $k = n-1 \rightarrow$  **Hamilton cycle problem**

## Application

Approximation algorithms for

- Graph-TSP
- Min. 2-edge connected subgraph



# Complexity: Max $C_{\leq k}$ -free 2-matching

	General Graph	Bipartite Graph
$k \geq n/2$	<b>NP-hard</b>	<b>NP-hard</b>
$k \geq 6$	<b>NP-hard</b> *1	<b>NP-hard</b> *3
$k = 5$	<b>NP-hard</b> *1	---
$k = 4$	<b>Open</b>	<b>P</b> *4
$k = 3$	<b>P</b> *2	---
$k = 2$	<b>P</b>	<b>P</b>

\*1: [Papadimitriou '78]

\*2: [Hartvigsen '84]

\*3: [Geelen '99]

\*4: [Hartvigsen '06], [Pap '07]

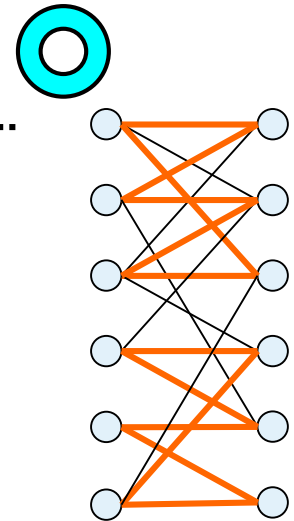
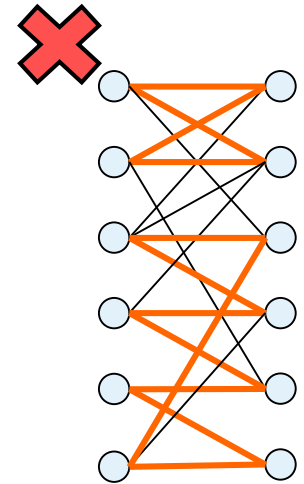
**$C_{\leq 4}$ -free 2-matching  
in bipartite graphs**

- **Well-solved**
- **Rich structure**



- $C_{\leq 4}$ -free 2-matchings in **Bipartite Graphs**:  
**Classical Matching Theory** is Extended

- **Min-max theorem** [Király '99][Frank '03]
  - König, Tutte-Berge
- **Combinatorial algorithm** [Hartvigsen '06][Pap '07]
  - Edmonds
- **Decomposition theorem** [T. '15]
  - Dulmage-Mendelsohn, Edmonds-Gallai



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- ◆ **Max. Weight  $C_{\leq 4}$ -free 2-matching**

- **NP-hard** in bipartite graphs
- Positive results for a certain class
  - **LP-formulation w/ dual integrality** [Makai '07]
    - Cunningham-Marsh
  - **Combinatorial algorithm** [T. '08]
  - **Discrete convexity** [Kobayashi, Szabó, T.: '12]

**Def.**  $A \subseteq \mathbb{Z}$

**2-factor**  $F$  is **A-covering**

def  
 $\Leftrightarrow F$  intersects every  $k$ -edge cut  $\forall k \in A$

● **Hamilton cycle** =  $\mathbb{Z}$ -covering 2-factor

● 2-edge connected cubic graph:

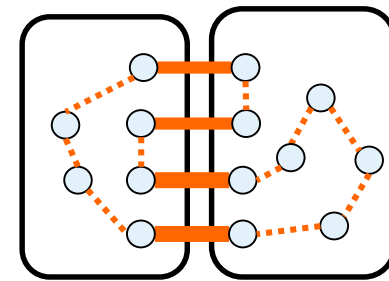
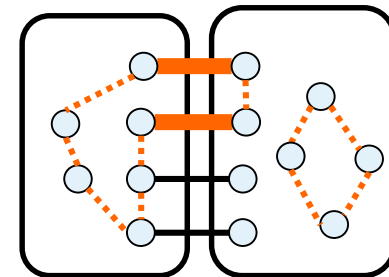
- **{3,4}-covering 2-factor** exists [Kaiser, Škrekovski '08],  
and can be found in  $O(n^3)$  time [Boyd, Iwata, T. '13]
- Min-weight **{3}-covering 2-factor** in  $O(n^3)$  time [BIT. '13]

● Graphs w/o **{4,5}-covering 2-factor**

[Čada, Chiba, Ozeki, Vrána, Yoshimoto '13]

➤ **Application:** Approximation of min. 2-edge connected subgraph

[BIT. '13]



- **Introduction:** 2-matching and Hamilton cycle
- **Previous Work**
  - Subtour Elimination
  - $C_{\leq k}$ -free 2-matching
  - $A$ -covering 2-factor
- **Our Framework:**  $\mathcal{U}$ -feasible 2-matching
  - Min-max Theorem
  - Combinatorial Algorithm
  - Decomposition Theorems
- **Conclusion**

# Our Framework: $\mathcal{U}$ -feasible 2-matching

$$\mathcal{U} = \{U_1, U_2, \dots\} \subseteq 2^V$$

**Def.**

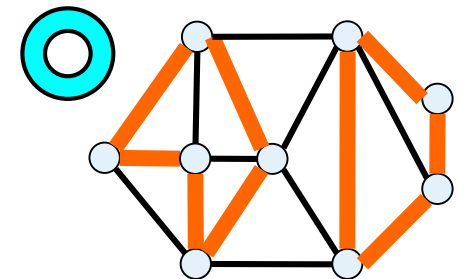
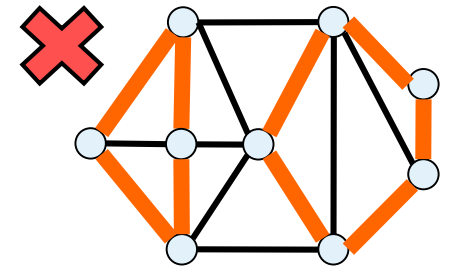
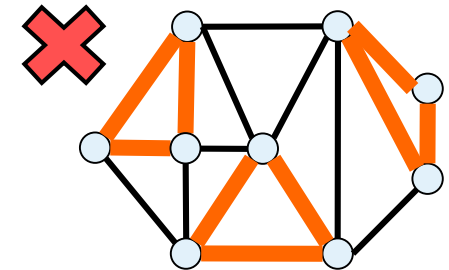
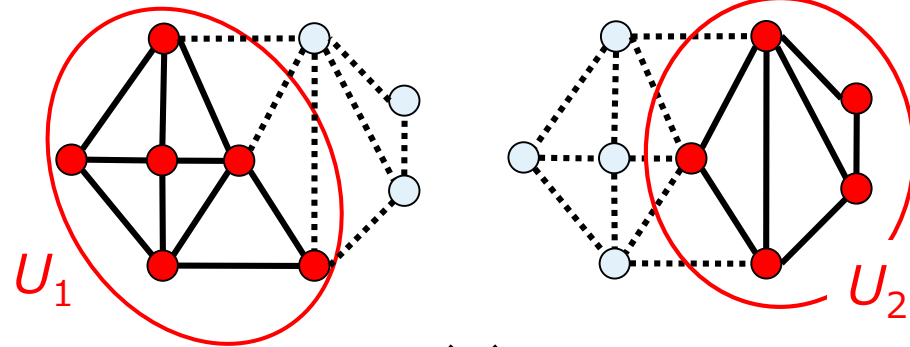
2-matching  $F$  is  **$\mathcal{U}$ -feasible**  
 $\Leftrightarrow |F[U]| \leq |U| - 1 \quad \forall U \in \mathcal{U}$

**Subtour Elimination** for  $U \in \mathcal{U}$

**Problem**

- Find a  **$\mathcal{U}$ -feasible 2-factor**
- Find a **max.  $\mathcal{U}$ -feasible 2-matching**

- **Hamilton cycle:**  $\mathcal{U} = 2^V \setminus \{V\}$
- **$C_{\leq k}$ -free 2-matching:**  $\mathcal{U} = \{U \subseteq V : |U| \leq k\}$
- **$\mathcal{A}$ -covering 2-factor:**  $\mathcal{U} = \{U \subseteq V : |\delta U| \in \mathcal{A}\}$



## Question

*When is the problem solved in poly. time ?*

### ● Known Facts:

- $\mathcal{U} = 2^V \setminus \{V\}$ : **NP-hard** [Hamilton cycle]
- $\mathcal{U} = \{U: |U| \leq 4\}$ , Bipartite: **P** [ $C_{\leq 4}$ -free 2-matching]
- $\mathcal{U} = \{U: |\delta U| \in \{3, 4\}\}$ , 2-EC Cubic: **P** [ $\{3, 4\}$ -covering 2-factor]

### ● Our Answer: **Extend the $C_{\leq 4}$ -free case**

- $\mathcal{U} = \{U: G[U] \text{ is Hamilton-laceable}\}$ , Bipartite

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### ◆ Polynomial in $n, m, \gamma$

- $\gamma$ : Time for determining  $\mathcal{U}$ -feasibility
- $|\mathcal{U}|$  might be exponential
- $\mathcal{U}$  is given implicitly

## Def.

[Simmons '78]

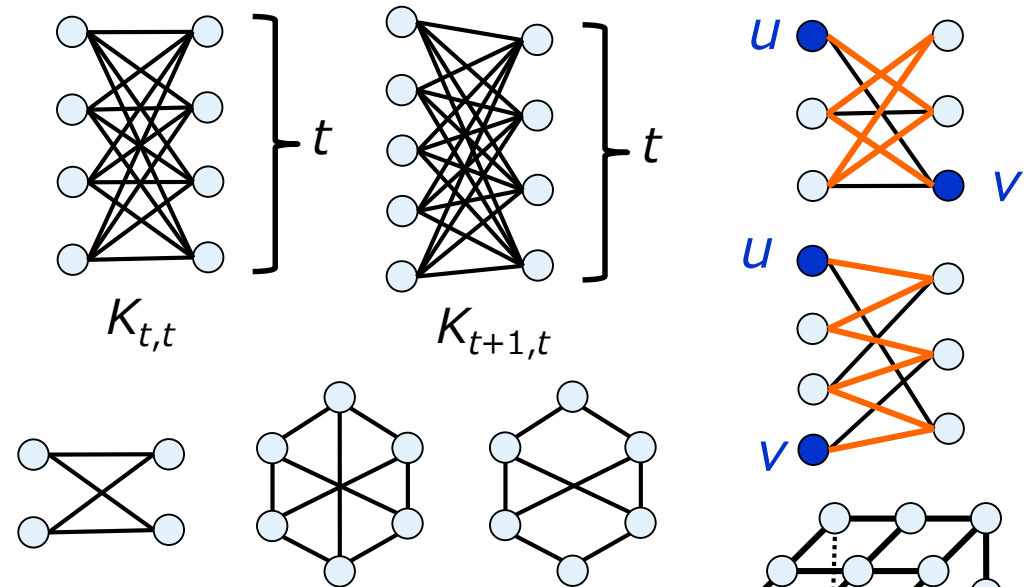
Bipartite graph  $G=(V^+,V^-;E)$  is a **Hamilton-Laceable**  $\Leftrightarrow$

- $|V^+|=|V^-|$ ,  $\forall u \in V^+, v \in V^-, \exists$  Hamilton path from  $u$  to  $v$
- $|V^+|=|V^-|+1$ ,  $\forall u, v \in V^+, \exists$  Hamilton path from  $u$  to  $v$

- $K_{t,t}$  or  $K_{t+1,t}$   
with  $\leq t-2$  edges deleted  
[Simmons '78]

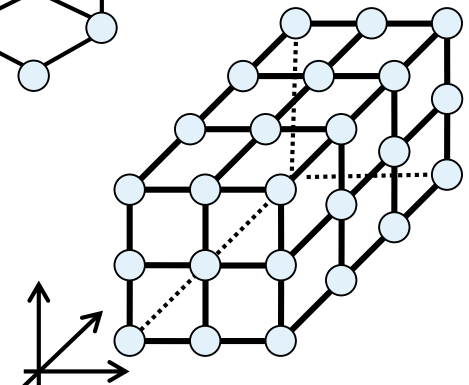
➤  $C_4$

➤  $C_6$  with  $\geq 2$  chords



- $d$ -dimensional rectangular lattice

- Except for  $2 \times r$ ,  $3 \times 2r$  ( $d=2$ ) [Simmons '81]
- $O(dn)$  edges



## Assumption

$G$ : **Bipartite**

$G[U]$  is **Hamilton-laceable**  $\forall U \in \mathcal{U}$

## Results

- **Min-max Theorem**
- **Combinatorial Algorithm**
- **Decomposition Theorems**

[Király '99]

[Hartvigsen '06][Pap '07]

[T. '15]



## Extension of $C_{\leq 4}$ -free 2-matching Theory

**Subtour Elimination** for  $U \in \mathcal{U}$

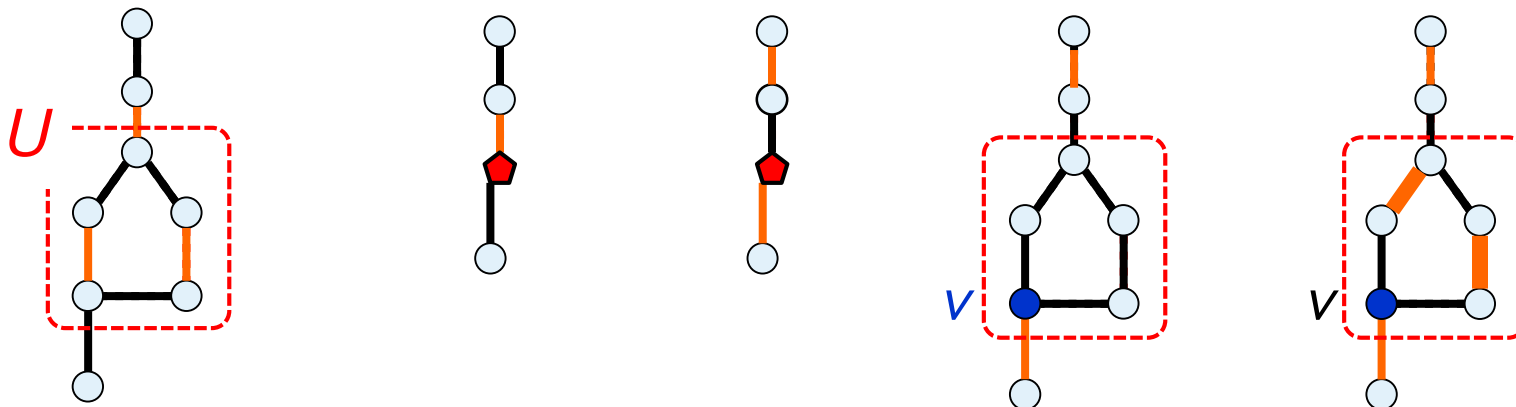
**Def.** 2-matching  $F$  is a  **$\mathcal{U}$ -feasible**  $\Leftrightarrow |F[U]| \leq |U| - 1 \forall U \in \mathcal{U}$

**Problem** Find a **max.  $\mathcal{U}$ -feasible 2-matching**

- Why Hamilton-laceable graphs?

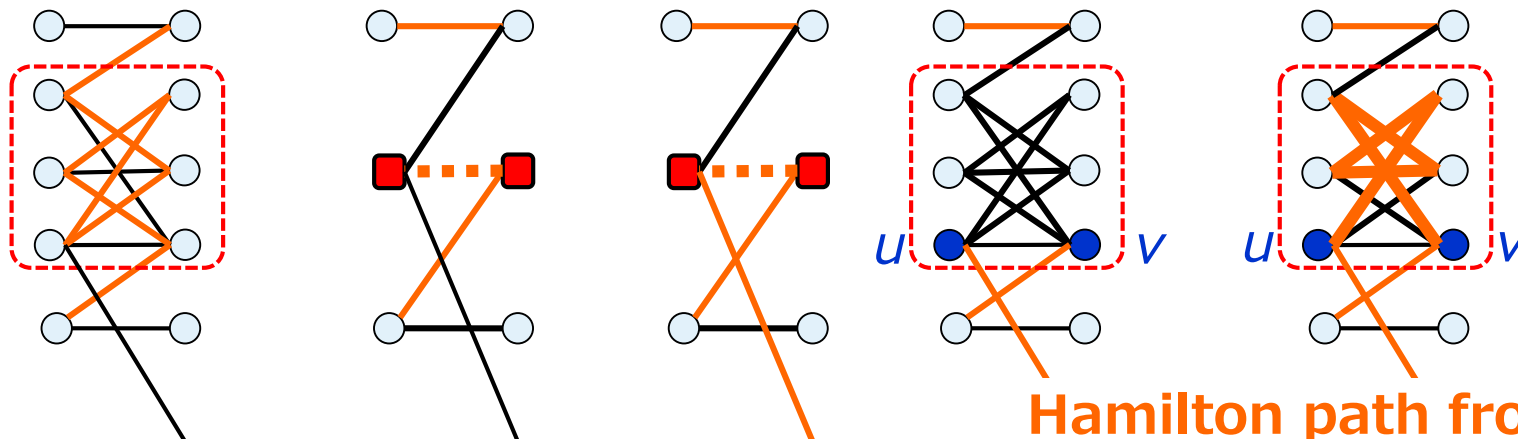
- Shrinking technique works!

- Nonbipartite matching : **Factor-critical graph** [Edmonds '65]



Matching covering  $U-v$

- Our problem: **Hamilton-laceable graph**



Hamilton path from  $u$  to  $v$



$G$ : Bipartite

## Theorem

If  $G[U]$  is Hamilton-laceable  $\forall U \in \mathcal{U}$ ,  
max.  **$\mathcal{U}$ -feasible** 2-matching is found in  **$O(n^3\gamma + n^2m)$  time**

If  $G[C]$  is Hamilton-laceable  $\forall$  cycle  $C$  of length  $\leq k$ ,  
max.  **$C_{\leq k}$ -free** 2-matching is found in  **$O(kn^3 + n^2m)$  time**

## Corollary

**Max. 2-matching** excluding  $C_4$  and  $C_6$  with  $\geq 2$  chords  
is found in  **$O(n^2m)$  time**

In every  **$d$ -regular bipartite graph** with  $d \geq 4$ ,  
a **2-factor** excluding  $C_4$  and  $C_6$  with  $\geq 2$  chords exists and  
can be found in  **$O(n^2m)$  time**

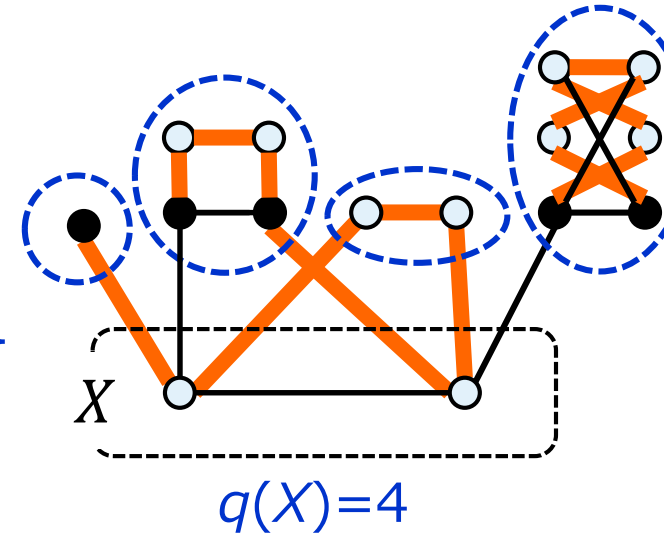
➤ **First positive results for  $C_{\leq 6}$ -free 2-matching**

## Theorem

$$\max \{ |F| : F \text{ is a } \mathcal{U}\text{-feasible 2-matching} \}$$

$$= \min \{ |V| + |X| - q(X) : X \subseteq V \}$$

$$q(X) = \#\{\text{Component } C \text{ in } G[\bar{X}] : V(C) \in \mathcal{U}\}$$



## Theorem [Király '99]

$$\max \{ |M| : M \text{ is a } C_4\text{-free 2-matching} \}$$

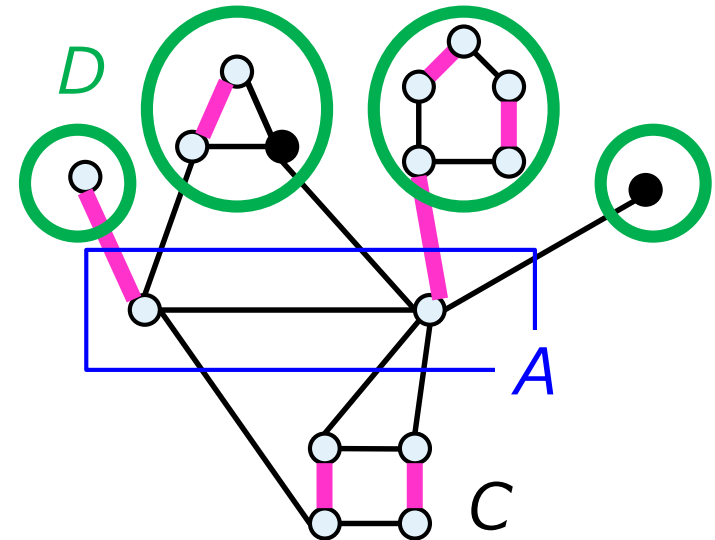
$$= \min \{ |V| + |X| - q_4(X) : X \subseteq V \}$$

$$q_4(X) = \#\{ \text{circles, } \text{---}, \text{ } \begin{array}{c} \circ \quad \circ \\ | \quad | \\ \circ \quad \circ \end{array} \text{ in } G[\bar{X}] \}$$

## Theorem [Tutte '47, Berge '58]

$$\max \{ |M| : M \text{ is a matching} \}$$

$$= \frac{1}{2} \min \{ |V| + |X| - \text{odd}(X) : X \subseteq V \}$$



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- **Conclusion**

## $\mathcal{U}$ -feasible 2-matching

- **New framework** of restricted 2-matching
- Excludes **subtours** for  $\forall U \in \mathcal{U} \subset 2^V$
- **Classical matching theory** is extended if
  - $G$  is **bipartite** and  $G[U]$  is **Hamilton-laceable**  $\forall U \in \mathcal{U}$

## Future Work

- **Weighted** version
  - LP, Algorithm, Discrete Convexity
- **Broader** solvable class than Hamilton-laceable graphs
- **Application**
  - Approximation for TSP etc.

