

Randomized Strategies for Cardinality Robustness in the Knapsack Problem

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Knapsack Problem

- Item set: E
- Profit: $p_e \geq 0$ ($e \in E$)
- Weight: $w_e \geq 0$ ($e \in E$)
- Capacity: $C \geq 0$

- Family of feas. sets $\mathcal{F} = \{X \subseteq E : w(X) \leq C\}$

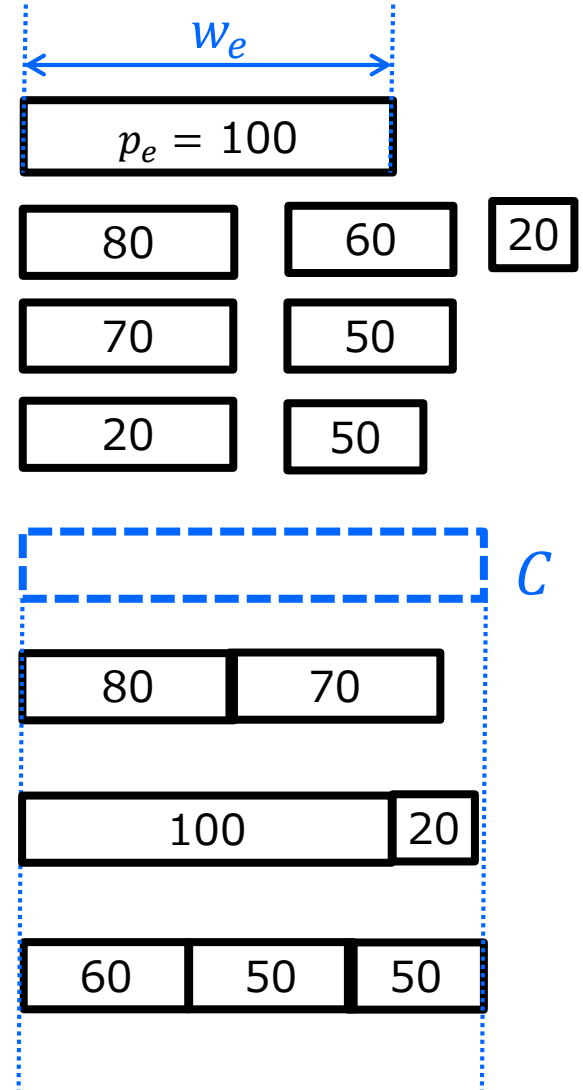
Problem

maximize $p(X)$
subject to $X \in \mathcal{F}$

$$w(X) = \sum_{e \in X} w_e$$

$$p(X) = \sum_{e \in X} p_e$$

- NP-hard
- FPTAS



- Cardinality constraint $|X| \leq k$ is given after choosing X
 - $X(k)$: expensive $\leq k$ items in X
 - OPT_k : optimal sol.

Def

$X \in \mathcal{F}, 0 < \alpha \leq 1$

➤ X is **α -robust** $\stackrel{\text{def}}{\iff} \forall k, p(X(k)) \geq \alpha \cdot p(OPT_k)$

➤ **robustness** $\stackrel{\text{def}}{=} \min_k \frac{p(X(k))}{p(OPT_k)}$

$$\underline{p(X(1)) = 80}$$

$$\underline{p(OPT_1) = 100}$$

$$p(X(2)) = 150$$

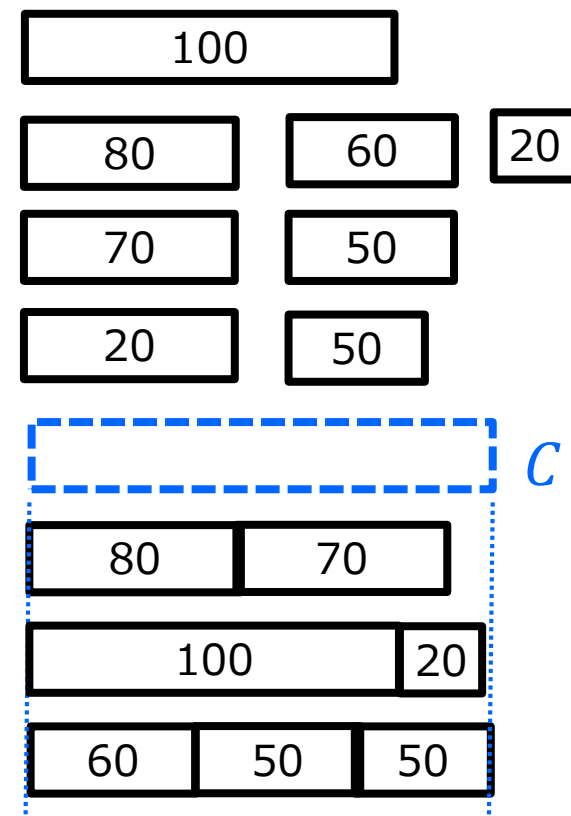
$$p(OPT_2) = 150$$

$$p(X(3)) = 150$$

$$p(OPT_3) = 160$$

⋮

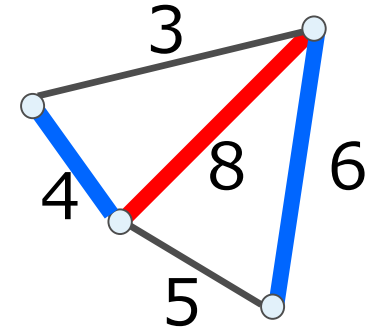
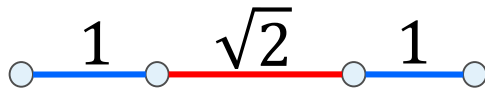
➔ **Robustness = 0.8**



- **Introduction:** Robust knapsack problem
- **Related Work**
 - [Hassin, Rubinstein \[2002\]](#): Robust matching
 - [Kakimura, Makino \[2013\]](#): Robust independence system
 - [Matuschke, Skutella, Soto \[2015\]](#): Mixed strategy
- **Our Result:** **Mixed strategy for robust knapsack problem**
 - Upper/Lower bound for robustness
 - Better than pure strategy
- **Concluding Remarks**

- **Hassin, Rubinfeld [2002]**

- **Matroid**: greedy alg. → **1-robust**
- **Matching**: maximizing $\sum_{e \in X} p_e^2$ → $\frac{1}{\sqrt{2}}$ -robust
- $\frac{1}{\sqrt{2}}$ is best possible



$$p(\text{OPT}_1) = 8$$

$$p(\text{OPT}_2) = 10$$

X : 0.8-robust

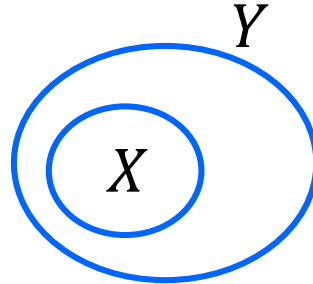
Y : 0.75-robust

- **Fujita, K, Makino [2013]**

- **Matroid Intersection**: maximizing $\sum_{e \in X} p_e^2$ → $\frac{1}{\sqrt{2}}$ -robust
- Computation of max robustness: **NP-hard**

Def

(E, \mathcal{F}) : independence system $\stackrel{\text{def}}{\Leftrightarrow} \begin{cases} \emptyset \in \mathcal{F}, \\ X \subseteq Y, Y \in \mathcal{F} \Rightarrow X \in \mathcal{F} \end{cases}$



• Kakimura, Makino [2013]

- **Ind. system**: maximizing $\sum_{e \in X} p_e^2 \rightarrow \frac{1}{\sqrt{\mu(\mathcal{F})}}$ -robust
- $\frac{1}{\sqrt{\mu(\mathcal{F})}}$ is best possible

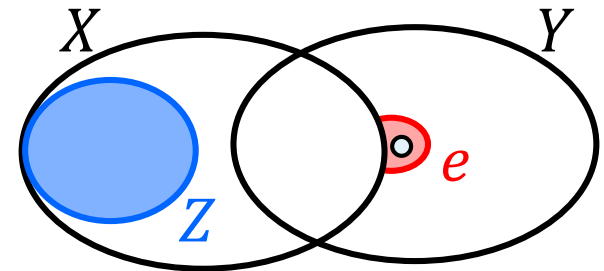
Def [Mestre 2006]

$\mu(\mathcal{F}) \triangleq$ min. integer μ satisfying

$X, Y \in \mathcal{F}, e \in Y - X$

$\Rightarrow \exists Z \subseteq X - Y$

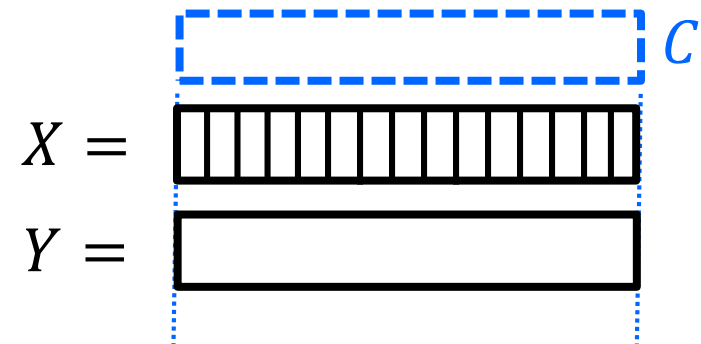
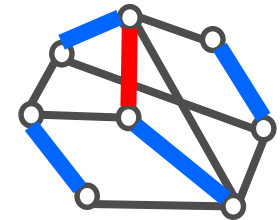
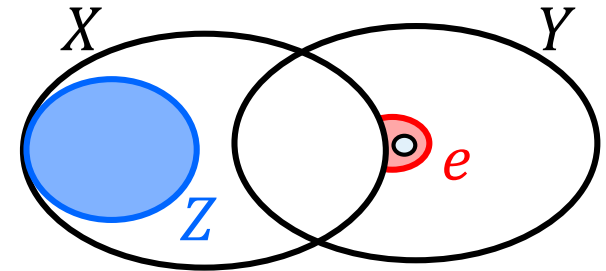
s.t. $|Z| \leq \mu, (X - Z) + e \in \mathcal{F}$



● **Kakimura, Makino [2013]:** max. $\sum_{e \in X} p_e^2 \rightarrow \frac{1}{\sqrt{\mu(\mathcal{F})}}$ -robust

- Matroid: $\mu(\mathcal{F}) = 1$
- Matching: $\mu(\mathcal{F}) \leq 2$
- Intersection of m matroids: $\mu(\mathcal{F}) \leq m$
- **Feasible sets of Knapsack Problem**
→ $\mu(\mathcal{F}) = M$ (arbitrarily large)

$$\begin{aligned} X &= \{e_1, \dots, e_M\} \quad (w_{e_i} = C/M) \\ Y &= \{e_0\} \quad \quad \quad (w_{e_0} = C) \end{aligned}$$



● **Kakimura, Makino, Seimi [2012]**

- Robust Knapsack Problem: weakly NP-hard + FPTAS

- Matuschke, Skutella, Soto [2015]: Zero-Sum game

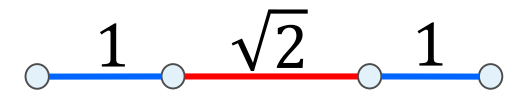
Alice: Choose $X \in \mathcal{F}$

Bob: Choose k (knowing X)

→ Alice's payoff = $\frac{p(X(k))}{p(\text{OPT}_k)}$

- Mixed Strategy = Distribution on \mathcal{F}
- Choose X_i with probability λ_i → **robustness**:

$$\min_k \mathbf{E} \left[\frac{p(X(k))}{p(\text{OPT}_k)} \right] = \min_k \frac{\sum_i \lambda_i p(X_i(k))}{p(\text{OPT}_k)}$$



$$p(\text{OPT}_1) = \sqrt{2}$$

$$p(\text{OPT}_2) = 2$$

➤ Ex. Choose X or Y with prob. $1/2$

$$\min \left\{ \frac{\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \sqrt{2}}{\sqrt{2}}, \frac{\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \sqrt{2}}{2} \right\} = \frac{2 + \sqrt{2}}{4} = 0.8535 \dots$$

Robustness of X, Y

$$\frac{1}{\sqrt{2}} = 0.7071 \dots$$

• Matuschke, Skutella, Soto [2015]

1. Choose x in $[0,1]$ uniformly at random
2. For each e , set $q_e := \log_2 p_e$, $p'_e := 2^{\lfloor q_e - x \rfloor}$, and find $X \in \mathcal{F}$ maximizing $p'(X)$

Round value p to power of two

Thm [MSS 15]

The above mixed strategy is $\frac{1}{\ln 4}$ -robust for

- Matching
- Matroid intersection
- Strongly base orderable matroid parity etc.

0.7213 ...

cf. $\frac{1}{\sqrt{2}} = 0.7071 ...$

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- **Our Result:** **Mixed strategy for robust knapsack problem**
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- **Concluding Remarks**

- Robustness of pure strategy: $\frac{1}{\sqrt{\mu(\mathcal{F})}}$ [Kakimura, Makino 13]

$\mu(\mathcal{F})$: arbitrarily large

- Robustness of mixed strategy [Our result]

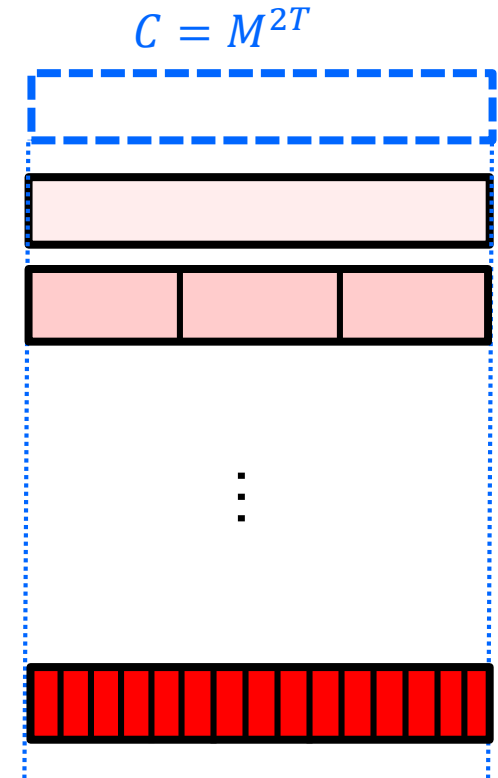
1. Upper bound $\mathcal{O}\left(\frac{\log \log \mu(\mathcal{F})}{\log \mu(\mathcal{F})}\right)$, $\mathcal{O}\left(\frac{\log \log \rho(\mathcal{F})}{\log \rho(\mathcal{F})}\right)$ $\rho(\mathcal{F})$: another parameter of ind. sys.

2. Lower bound $\Omega\left(\frac{1}{\log \mu(\mathcal{F})}\right)$, $\Omega\left(\frac{1}{\log \rho(\mathcal{F})}\right)$: Design a strategy

↳ Extend to ind. sys.: $\mathcal{O}\left(\frac{1}{\log \mu(\mathcal{F})}\right)$, $\mathcal{O}\left(\frac{1}{\log \rho(\mathcal{F})}\right)$, $\Omega\left(\frac{1}{\log \rho(\mathcal{F})}\right)$

Result 1. Upper Bound: Hard Instance

Type	w_e	p_e	Number	p_e/w_e	Total profit
0	M^{2T}	M^{2T}	1	1	M^{2T}
1	M^{2T-2}	M^{2T-1}	M^2	M	M^{2T+1}
:	:	:	:	:	:
i	M^{2T-2i}	M^{2T-i}	M^{2i}	M^i	M^{2T+i}
:	:	:	:	:	:
T	1	M^T	M^{2T}	M^T	M^{3T}



$$p(\text{OPT}(1)) = M^{2T}, \quad p(\text{OPT}(M^{2T})) = M^{3T}$$

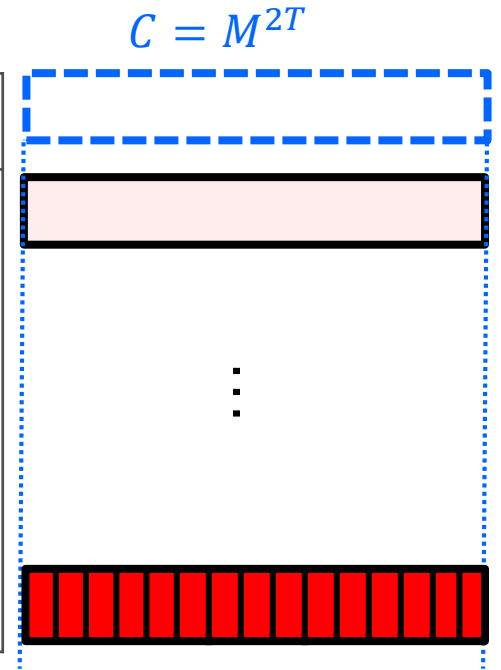
Thm [Our result]

For any mixed strategy, robustness $\leq \frac{1}{T+1} + \frac{2}{M}$

➤ No mixed strategy can achieve constant robustness

Result 1. Upper Bound: Hard Instance

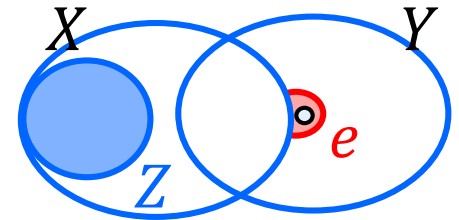
Type	w_e	p_e	Number	p_e/w_e	Total profit
0	M^{2T}	M^{2T}	1	1	M^{2T}
:	:	:	:	:	:
i	M^{2T-2i}	M^{2T-i}	M^{2i}	M^i	M^{2T+i}
:	:	:	:	:	:
T	1	M^T	M^{2T}	M^T	M^{3T}



$\rightarrow \mu(\mathcal{F}) = M^{2T}$

Thm [Our result]

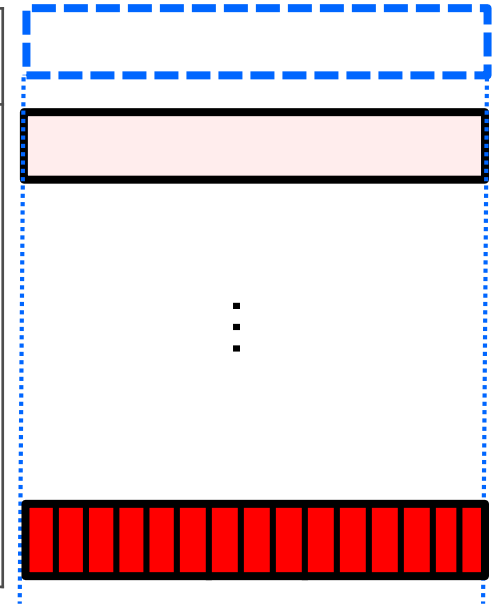
For any mixed strategy, robustness $\leq \frac{1}{T+1} + \frac{2}{M}$



Result 1. Upper Bound: Hard Instance

$$C = M^{2T}$$

Type	w_e	p_e	Number	p_e/w_e	Total profit
0	M^{2M}	M^{2M}	1	1	M^{2M}
:	:	:	:	:	:
i	M^{2M-2i}	M^{2M-i}	M^{2i}	M^i	M^{2M+i}
:	:	:	:	:	:
M	1	M^M	M^{2M}	M^M	M^{3M}



$$\rightarrow \mu(\mathcal{F}) = M^{2M}$$

$$\log M^{2M} = \Theta(M \log M)$$

$$\log \log M^{2M} = \Theta(\log M)$$

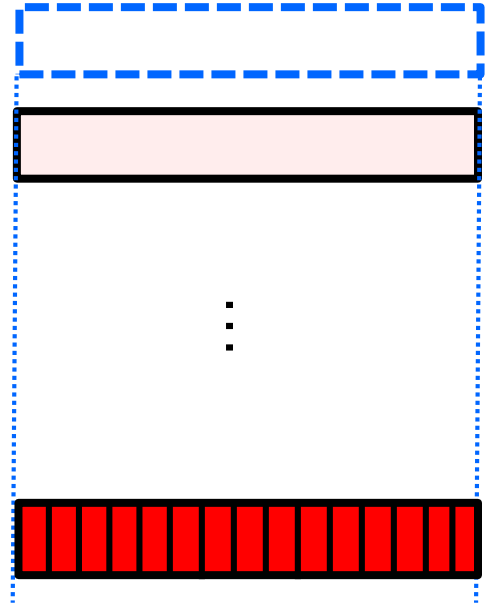
Thm [Our result]

For any mixed strategy, robustness $\leq \frac{3}{M}$

Result 1. Upper Bound: Hard Instance

$$C = M^{2T}$$

Type	w_e	p_e	Number	p_e/w_e	Total profit
0	M^{2M}	M^{2M}	1	1	M^{2M}
:	:	:	:	:	:
i	M^{2M-2i}	M^{2M-i}	M^{2i}	M^i	M^{2M+i}
:	:	:	:	:	:
M	1	M^M	M^{2M}	M^M	M^{3M}



$$\rightarrow \mu(\mathcal{F}) = M^{2M}$$

$$\log M^{2M} = \Theta(M \log M)$$

$$\log \log M^{2M} = \Theta(\log M)$$

Thm [Our result]

For any mixed strategy, robustness $\leq \frac{3}{M} = o\left(\frac{\log \log \mu(\mathcal{F})}{\log \mu(\mathcal{F})}\right)$

Result 2. Lower Bound: $\Omega(1/\log \mu(\mathcal{F}))$

Strategy (A)

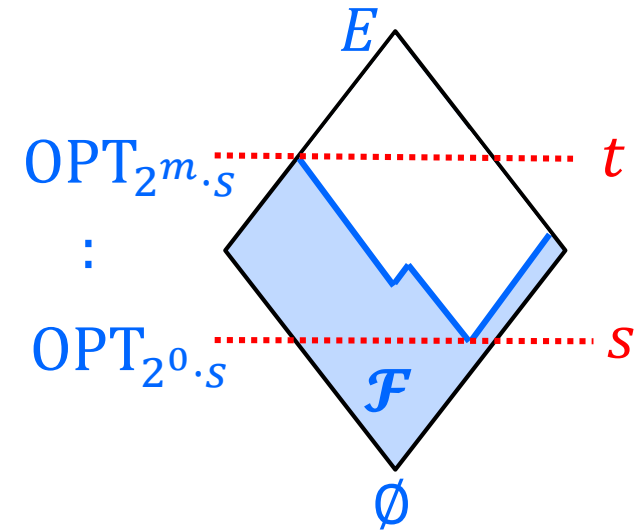
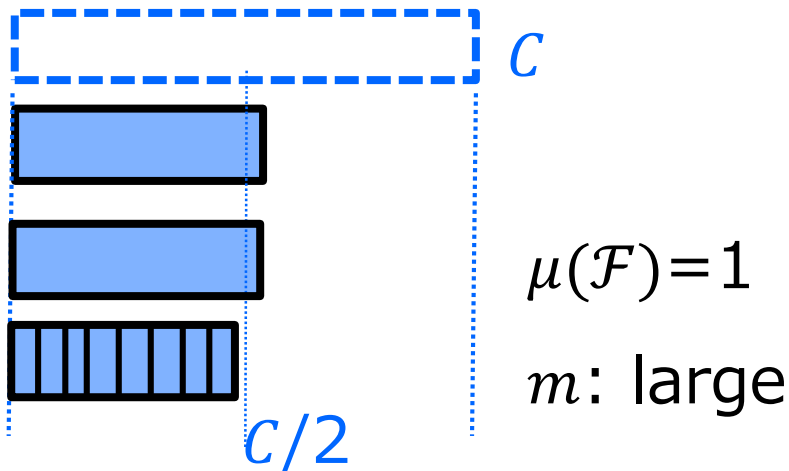
$$m := \lceil \log (t/s) \rceil$$

- $\forall i \in \{0, 1, \dots, m\}$, choose $X_i = \text{OPT}_{2^i \cdot s}$ with prob. $\frac{1}{m+1}$

Thm [Our result]

$$\text{Robustness} \geq \frac{1}{m+1}$$

$m = O(\log \mu(\mathcal{F}))$??? \rightarrow NO

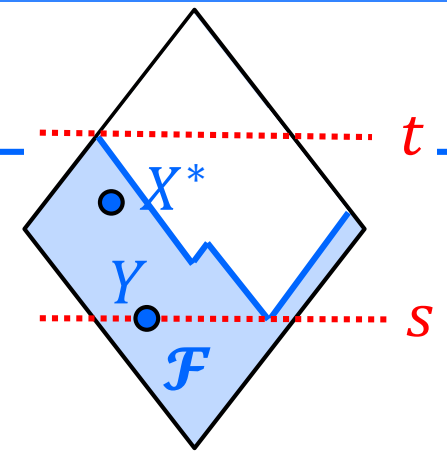


Idea

Choose small items **in advance**

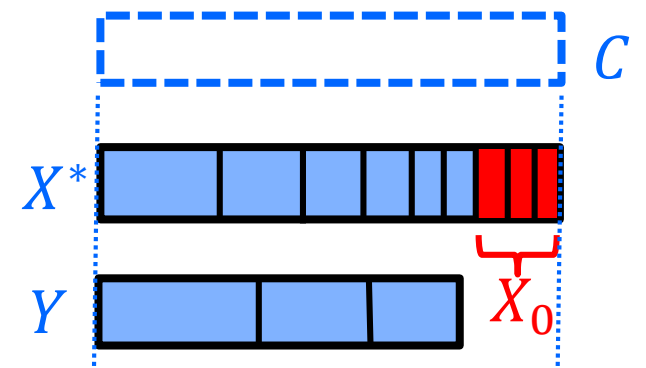
Strategy (B)

1. X^* : optimal sol., Y : heaviest s elements
2. $X_0 \subseteq X^*$: $w(X_0) \leq C - w(Y)$ with max size
3. $C' := C - w(X_0)$, $E' := E - X_0$, $m' := \left\lceil \frac{\log |X^* - X_0|}{s} \right\rceil$
4. $\forall i \in \{0, 1, \dots, m'\}$, choose $\text{OPT}'_{2^{i \cdot s}} \cup X_0$ with prob. $\frac{1}{m'+1}$



Thm [Our result]

$$\text{Robustness} \geq \frac{1}{4(m'+1)} = \Omega\left(\frac{1}{\log \mu(\mathcal{F})}\right)$$



cf. pure strategy: $\frac{1}{\sqrt{\mu(\mathcal{F})}}$ [Kakimura, Makino 13]

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● Our result: mixed strategy for robust knapsack problem

1. Upper bound $\mathcal{O}\left(\frac{\log \log \mu(\mathcal{F})}{\log \mu(\mathcal{F})}\right), \mathcal{O}\left(\frac{\log \log \rho(\mathcal{F})}{\log \rho(\mathcal{F})}\right)$

2. Lower bound $\Omega\left(\frac{1}{\log \mu(\mathcal{F})}\right), \Omega\left(\frac{1}{\log \rho(\mathcal{F})}\right)$: Design a strategy

↳ Extend to ind. sys.: $\mathcal{O}\left(\frac{1}{\log \mu(\mathcal{F})}\right), \mathcal{O}\left(\frac{1}{\log \rho(\mathcal{F})}\right), \Omega\left(\frac{1}{\log \rho(\mathcal{F})}\right)$

● Future work

1. Close the gap between upper and lower bounds

2. $\Omega\left(\frac{1}{\log \mu(\mathcal{F})}\right)$ -robust strategy for general ind. sys.

3. Evaluation with rank quotient $r(\mathcal{F})$

$$r(\mathcal{F}) := \min_{X \subseteq E} \frac{\min\{|\text{maximal sol in } X|\}}{\max\{|\text{maximal sol in } X|\}}$$