Randomized Strategies for Cardinality Robustness in the Knapsack Problem

Yusuke Kobayashi University of Tsukuba
Kenjiro Takazawa Kyoto University

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Knapsack Problem

- Item set: E
- Profit: $p_e \geq 0 \ (e \in E)$
- Weight: $w_e \geq 0 \ (e \in E)$
- Capacity: $C \geq 0$

- Family of feas. sets $\mathcal{F} = \{X \subseteq E : w(X) \leq C\}$

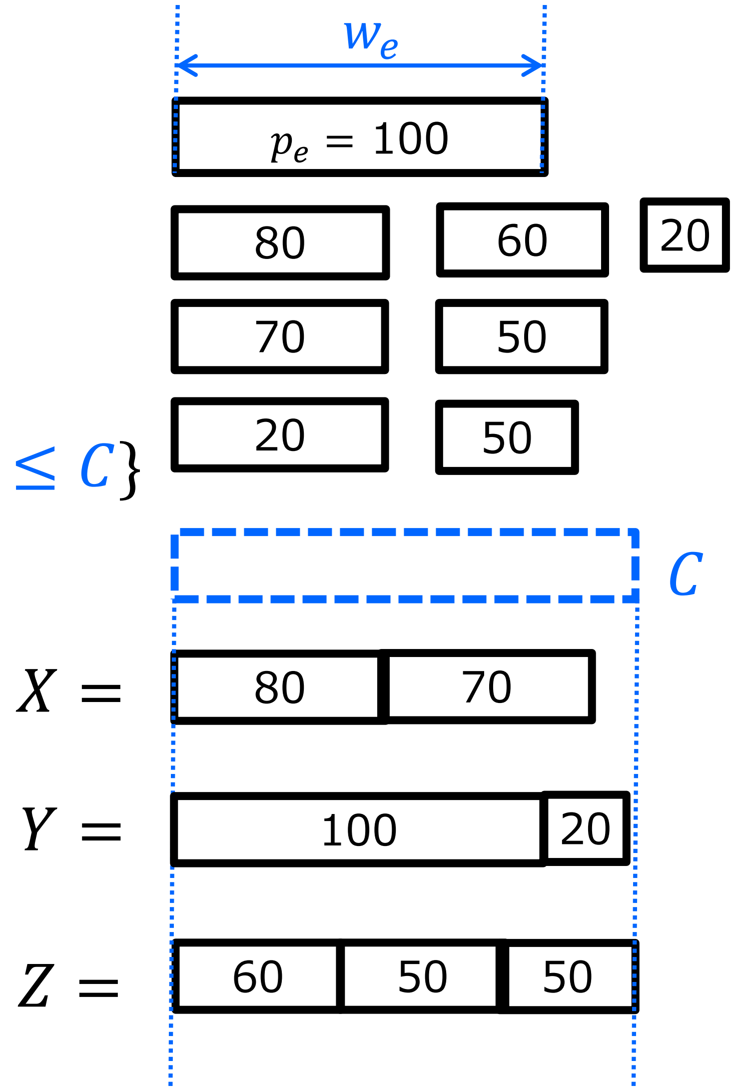
$$w(X) = \sum_{e \in X} w_e$$

- Problem

maximize $p(X)$
subject to $X \in \mathcal{F}$

$$p(X) = \sum_{e \in X} p_e$$

➢ NP-hard
➢ FPTAS



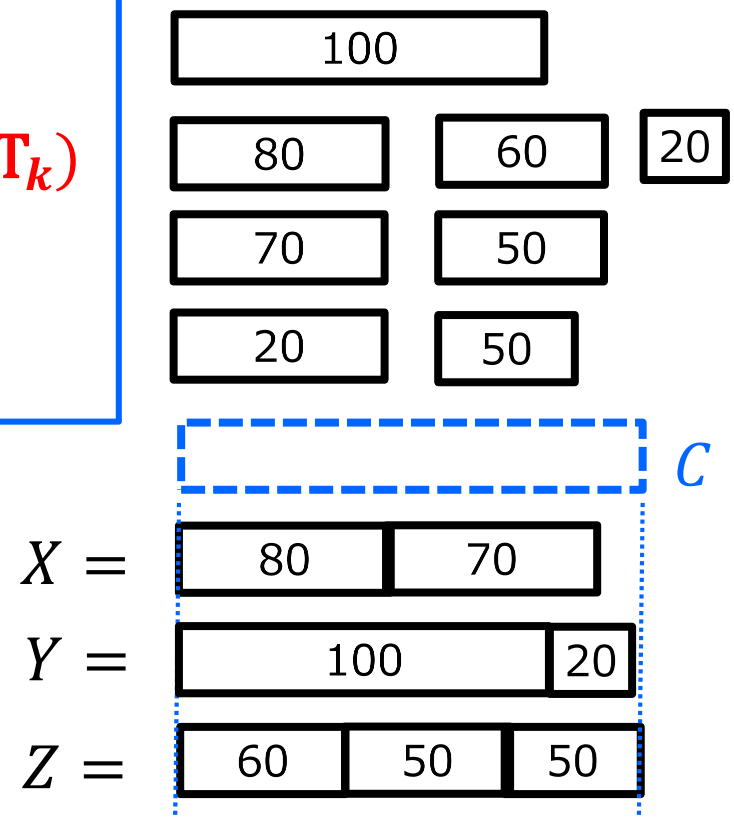
Cardinality Robustness

- Cardinality constraint $|X| \leq k$ is given after choosing X
  - $X(k)$: expensive $\leq k$ items in X
  - OPT_k : optimal sol.

Def

$X \in \mathcal{F}, 0 < \alpha \leq 1$

- X is ***α*-robust** $\overset{\text{def}}{\Longleftrightarrow} \forall k,\ p(X(k)) \geq \alpha \cdot p(\mathrm{OPT}_k)$
- **robustness** $\overset{\text{def}}{=} \min_k \dfrac{p(X(k))}{p(\mathrm{OPT}_k)}$



$p(X(1)) = 80$ $\qquad$ $p(\mathrm{OPT}_1) = 100$

$p(X(2)) = 150$ $\qquad$ $p(\mathrm{OPT}_2) = 150$

$p(X(3)) = 150$ $\qquad$ $p(\mathrm{OPT}_3) = 160$

$\vdots$

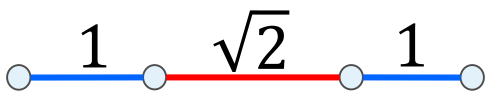
➔ **Robustness = 0.8**

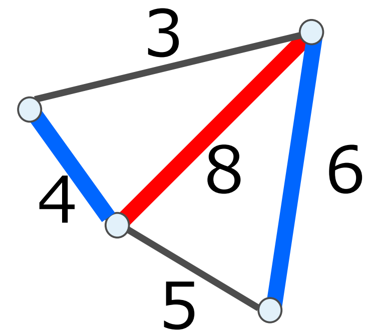
Contents

- **Introduction**: Robust knapsack problem
- **Related Work**
  - Hassin, Rubinstein [2002]: Robust matching
  - Kakimura, Makino [2013]: Robust independence system
  - Matuschke, Skutella, Soto [2015]: Mixed strategy
- **Our Result**: Mixed strategy for robust knapsack problem
  - Upper/Lower bound for robustness
  - Better than pure strategy
- **Concluding Remarks**

Matching / Matroid Intersection

- **Hassin, Rubinstein [2002]**
  - **Matroid**: greedy alg. ➔ **1-robust**
  - **Matching**: maximizing $\sum_{e \in X} p_e^2$ ➔ $\frac{1}{\sqrt{2}}$**-robust**
  - $\frac{1}{\sqrt{2}}$ is best possible





$p(\mathrm{OPT}_1) = 8$

$p(\mathrm{OPT}_2) = 10$

X: 0.8-robust

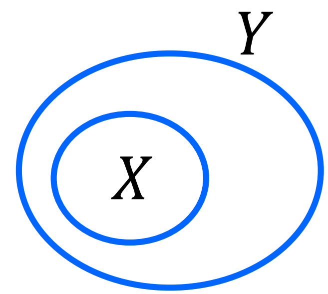
Y: 0.75-robust

- **Fujita, K, Makino [2013]**
  - **Matroid Intersection**: maximizing $\sum_{e \in X} p_e^2$ ➔ $\frac{1}{\sqrt{2}}$**-robust**
  - Computation of max robustness: **NP-hard**

Robust Independence System

Def

$(E, \mathcal{F})$: independence system $\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \emptyset \in \mathcal{F}, \\ X \subseteq Y, Y \in \mathcal{F} \Rightarrow X \in \mathcal{F} \end{cases}$

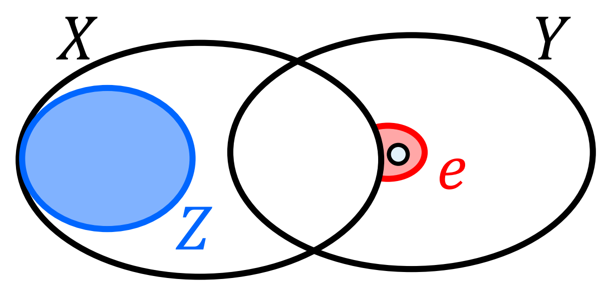


- **Kakimura, Makino [2013]**
  - **Ind. system**: maximizing $\sum_{e \in X} p_e^2$ ➔ $\frac{1}{\sqrt{\mu(\mathcal{F})}}$**-robust**
  - $\frac{1}{\sqrt{\mu(\mathcal{F})}}$ is best possible

Def [Mestre 2006]

$\mu(\mathcal{F}) \triangleq$ min. integer μ satisfying

$$X, Y \in \mathcal{F}, e \in Y - X$$
$$\Rightarrow \exists Z \subseteq X - Y$$
$$\text{s.t. } |Z| \leq \mu, (X - Z) + e \in \mathcal{F}$$



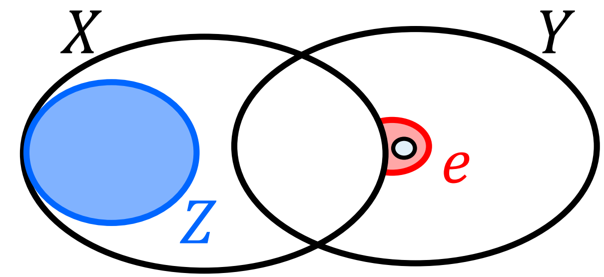
$\mu(\mathcal{F})$: Tractability of Independence System

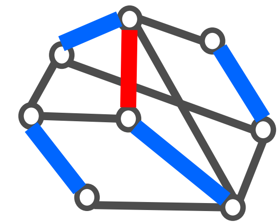
- **Kakimura, Makino [2013]:** max. $\sum_{e \in X} p_e^2$ ➔ $\frac{1}{\sqrt{\mu(\mathcal{F})}}$**-robust**

➢ Matroid: $\mu(\mathcal{F}) = 1$

➢ Matching: $\mu(\mathcal{F}) \leq 2$

➢ Intersection of m matroids: $\mu(\mathcal{F}) \leq m$

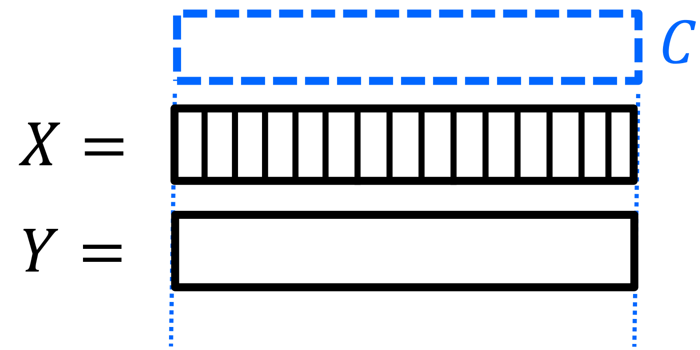




➢ **Feasible sets of Knapsack Problem**

➔ $\mu(\mathcal{F}) = M$ (arbitrarily large)

$$X = \{e_1, \dots, e_M\} \; \left(w_{e_i} = C/M\right)$$
$$Y = \{e_0\} \qquad (w_{e_0} = C)$$



- **Kakimura, Makino, Seimi [2012]**

➢ Robust Knapsack Problem: weakly NP-hard + FPTAS

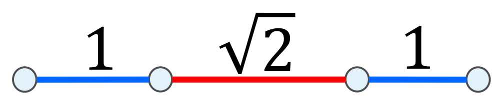
Mixed (or Randomized) Strategy

- **Matuschke, Skutella, Soto [2015]**: **Zero-Sum game**

> Alice: Choose $X \in \mathcal{F}$
>
> Bob: Choose k (knowing X)
>
> ➔ Alice's payoff = $\dfrac{p(X(k))}{p(\mathrm{OPT}_k)}$

- Mixed Strategy = Distribution on $\mathcal{F}$
- Choose X_i with probability λ_i ➔ **robustness**:

$$\min_k \mathbf{E}\left[\frac{p(X(k))}{p(\mathrm{OPT}_k)}\right] = \min_k \frac{\sum_i \lambda_i\, p(X_i(k))}{p(\mathrm{OPT}_k)}$$



$p(\mathrm{OPT}_1) = \sqrt{2}$

$p(\mathrm{OPT}_2) = 2$

- Ex. Choose X or Y with prob. ½

$$\min\left\{\frac{\frac{1}{2}\cdot 1+\frac{1}{2}\cdot\sqrt{2}}{\sqrt{2}},\ \frac{\frac{1}{2}\cdot 2+\frac{1}{2}\cdot\sqrt{2}}{2}\right\} = \frac{2+\sqrt{2}}{4} = 0.8535\ldots$$

> Robustness of X, Y
>
> $\frac{1}{\sqrt{2}} = 0.7071\ldots$

Mixed (or Randomized) Strategy

- **Matuschke, Skutella, Soto [2015]**

1. Choose x in $[0,1]$ uniformly at random
2. For each e, set $q_e \coloneqq \log_2 p_e$, $p'_e \coloneqq 2^{\lfloor q_e - x \rfloor}$, and find $X \in \mathcal{F}$ maximizing $p'(X)$

Round value p to power of two

Thm [MSS 15]

The above mixed strategy is $\frac{1}{\ln 4}$-robust for

- Matching
- Matroid intersection
- Strongly base orderable matroid parity etc.

0.7213 ...

cf. $\frac{1}{\sqrt{2}} = 0.7071 \ldots$

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Mixed Strategy for Robust Knapsack Problem

- Robustness of pure strategy: $\frac{1}{\sqrt{\mu(\mathcal{F})}}$ [Kakimura, Makino 13]

  $\mu(\mathcal{F})$: arbitrarily large

- **Robustness of mixed strategy [Our result]**

  1. **Upper bound** $O\left(\frac{\log\log\mu(\mathcal{F})}{\log\mu(\mathcal{F})}\right)$, $O\left(\frac{\log\log\rho(\mathcal{F})}{\log\rho(\mathcal{F})}\right)$

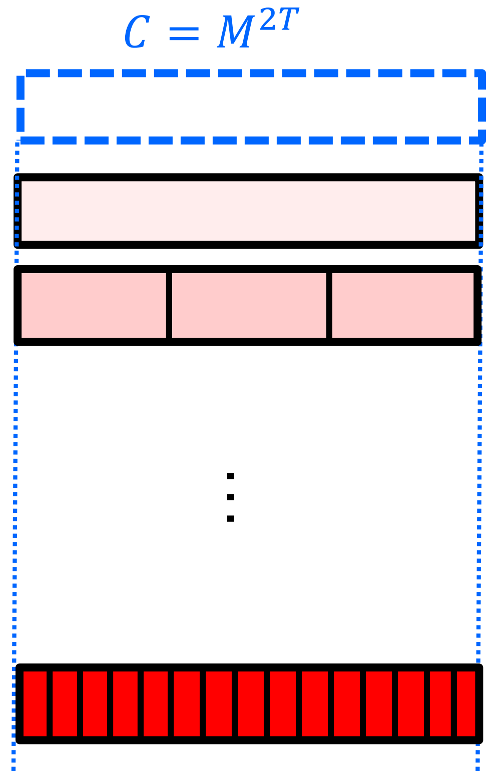
     $\rho(\mathcal{F})$: another parameter of ind. sys.

  2. **Lower bound** $\Omega\left(\frac{1}{\log\mu(\mathcal{F})}\right)$, $\Omega\left(\frac{1}{\log\rho(\mathcal{F})}\right)$: Design a strategy

  ⇒ **Extend to ind. sys.**: $O\left(\frac{1}{\log\mu(\mathcal{F})}\right)$, $O\left(\frac{1}{\log\rho(\mathcal{F})}\right)$, $\Omega\left(\frac{1}{\log\rho(\mathcal{F})}\right)$

Result 1. Upper Bound: Hard Instance

Type	w_e	p_e	Number	p_e/w_e	Total profit
0	M^{2T}	M^{2T}	1	1	M^{2T}
1	M^{2T-2}	M^{2T-1}	M^2	M	M^{2T+1}
⋮	⋮	⋮	⋮	⋮	⋮
i	M^{2T-2i}	M^{2T-i}	M^{2i}	M^i	M^{2T+i}
⋮	⋮	⋮	⋮	⋮	⋮
T	1	M^T	M^{2T}	M^T	M^{3T}



$$p(\mathrm{OPT}(1)) = M^{2T},\ p(OPT(M^{2T})) = M^{3T}$$

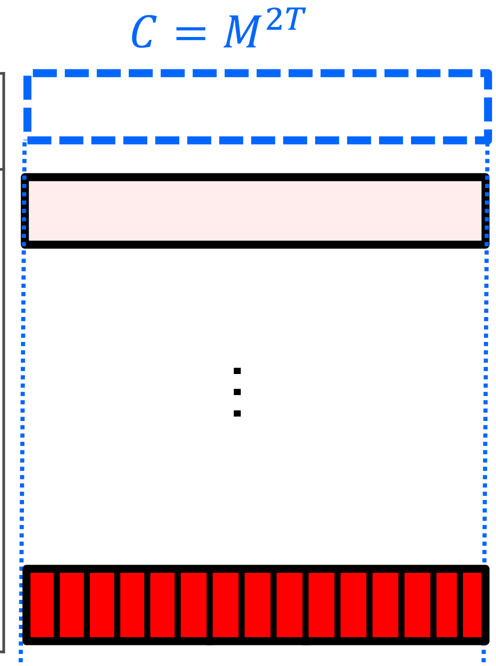
Thm [Our result]

For any mixed strategy, robustness $\leq \frac{1}{T+1} + \frac{2}{M}$

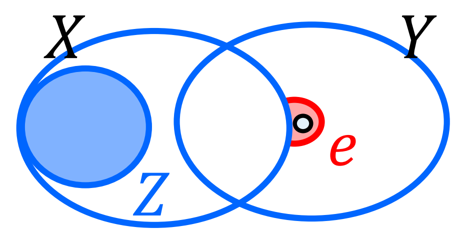
➢ **No mixed strategy can achieve constant robustness**

Result 1. Upper Bound: Hard Instance

Type	w_e	p_e	Number	p_e/w_e	Total profit
0	M^{2T}	M^{2T}	1	1	M^{2T}
⋮	⋮	⋮	⋮	⋮	⋮
i	M^{2T-2i}	M^{2T-i}	M^{2i}	M^i	M^{2T+i}
⋮	⋮	⋮	⋮	⋮	⋮
T	1	M^T	M^{2T}	M^T	M^{3T}



➔ $\boldsymbol{\mu}(\mathcal{F}) = \boldsymbol{M^{2T}}$

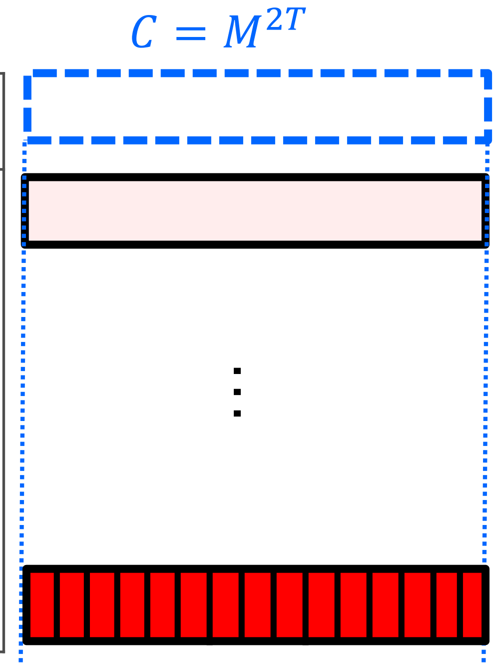


Thm [Our result]

For any mixed strategy, robustness $\leq \frac{\mathbf{1}}{\boldsymbol{T+1}} + \frac{\mathbf{2}}{\boldsymbol{M}}$

Result 1. Upper Bound: Hard Instance

Type	w_e	p_e	Number	p_e/w_e	Total profit
0	M^{2M}	M^{2M}	1	1	M^{2M}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	M^{2M-2i}	M^{2M-i}	M^{2i}	M^i	M^{2M+i}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
M	1	M^M	M^{2M}	M^M	M^{3M}



$\Rightarrow \boldsymbol{\mu}(\mathcal{F}) = \boldsymbol{M^{2M}}$

$$\log M^{2M} = \Theta(M \log M)$$

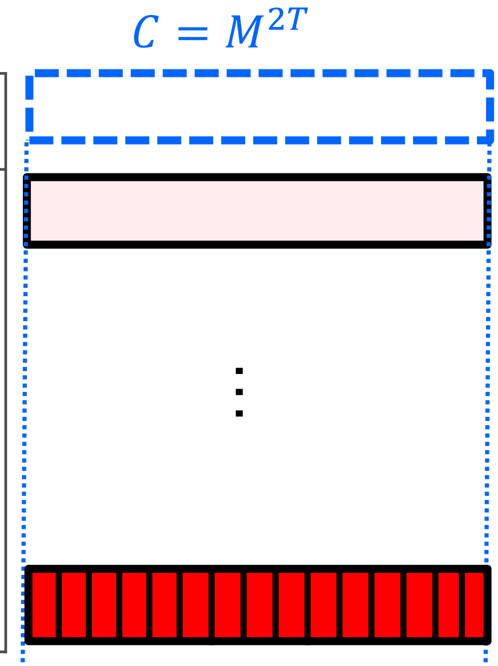
$$\log \log M^{2M} = \Theta(\log M)$$

Thm [Our result]

For any mixed strategy, robustness $\leq \frac{3}{M}$

Result 1. Upper Bound: Hard Instance

Type	w_e	p_e	Number	p_e/w_e	Total profit
0	M^{2M}	M^{2M}	1	1	M^{2M}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	M^{2M-2i}	M^{2M-i}	M^{2i}	M^{i}	M^{2M+i}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
M	1	M^{M}	M^{2M}	M^{M}	M^{3M}



$\Rightarrow$ $\boldsymbol{\mu(\mathcal{F}) = M^{2M}}$

$$\log M^{2M} = \Theta(M \log M)$$
$$\log\log M^{2M} = \Theta(\log M)$$

Thm [Our result]

For any mixed strategy, robustness $\leq \frac{3}{M} = O\left(\frac{\log\log\mu(\mathcal{F})}{\log\mu(\mathcal{F})}\right)$

Result 2. Lower Bound: $\Omega\left({}^{1}\!/_{\log \mu(\mathcal{F})}\right)$

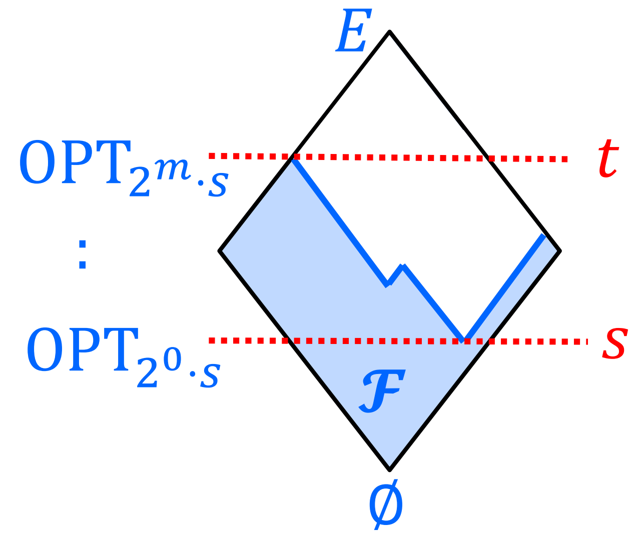
$m \coloneqq \lceil \log ({}^{t}\!/_{s}) \rceil$

Strategy (A)

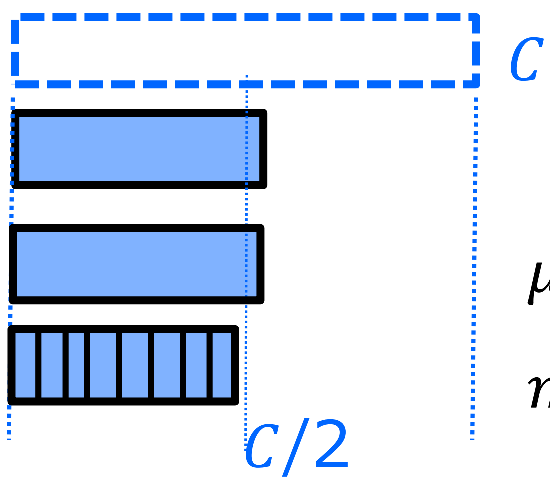
- $\forall i \in \{0,1,\ldots,m\}$, choose $\boldsymbol{X_i = \mathrm{OPT}_{2^i \cdot s}}$ with prob. $\frac{1}{m+1}$

Thm [Our result]

Robustness $\geq \frac{1}{m+1}$



$m = O(\log \mu(\mathcal{F}))$??? ➔ NO



$\mu(\mathcal{F}) = 1$

m: large

$\boldsymbol{t} \coloneqq \max\{|X| : X \in \mathcal{F}\}$

$\boldsymbol{s} \coloneqq \min\{|X| : X \notin \mathcal{F}\} - 1$

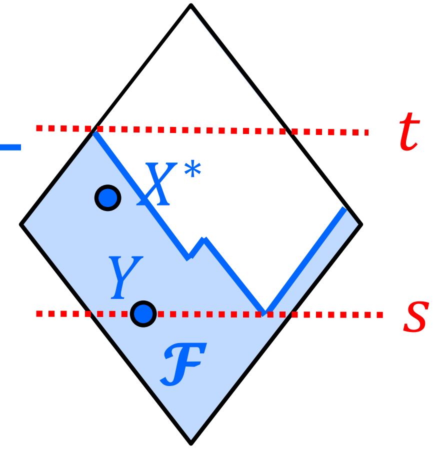
Idea

Choose small items in advance

Result 2. Lower Bound: $\Omega\left({}^{1}/_{\log \mu(\mathcal{F})}\right)$

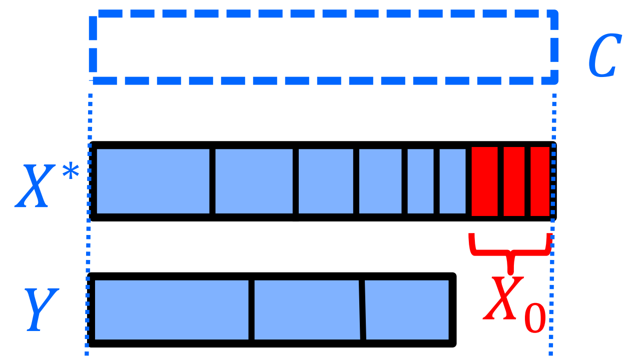
Strategy (B)

1. X^*: optimal sol., Y: heaviest s elements
2. $X_0 \subseteq X^*$: $w(X_0) \le C - w(Y)$ with max size
3. $C' := C - w(X_0)$, $E' := E - X_0$, $m' := \left\lceil \frac{\log|X^* - X_0|}{s} \right\rceil$
4. $\forall i \in \{0,1,\ldots,m'\}$, choose $\mathbf{OPT'}_{2^i \cdot s} \cup \boldsymbol{X_0}$ with prob. $\frac{1}{m'+1}$



Thm [Our result]

Robustness $\ge \frac{1}{4(m'+1)} = \Omega\left(\frac{1}{\log \mu(\mathcal{F})}\right)$



cf. pure strategy: $\frac{1}{\sqrt{\mu(\mathcal{F})}}$ [Kakimura, Makino 13]

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Summary・Future work

- **Our result: mixed strategy for robust knapsack problem**
  1. **Upper bound** $\mathbf{O}\left(\frac{\log\log\mu(\mathcal{F})}{\log\mu(\mathcal{F})}\right)$, $\mathbf{O}\left(\frac{\log\log\rho(\mathcal{F})}{\log\rho(\mathcal{F})}\right)$
  2. **Lower bound** $\mathbf{\Omega}\left(\frac{1}{\log\mu(\mathcal{F})}\right)$, $\mathbf{\Omega}\left(\frac{1}{\log\rho(\mathcal{F})}\right)$: Design a strategy

     ↳ **Extend to ind. sys.**: $\mathbf{O}\left(\frac{1}{\log\mu(\mathcal{F})}\right)$, $\mathbf{O}\left(\frac{1}{\log\rho(\mathcal{F})}\right)$, $\mathbf{\Omega}\left(\frac{1}{\log\rho(\mathcal{F})}\right)$

- **Future work**
  1. Close the gap between upper and lower bounds
  2. $\Omega\left(\frac{1}{\log\mu(\mathcal{F})}\right)$-robust strategy for general ind. sys.
  3. Evaluation with rank quotient $r(\mathcal{F})$

$$r(\mathcal{F}) := \min_{X \subseteq E} \frac{\min\{|\text{maximal sol in } X|\}}{\max\{|\text{maximal sol in } X|\}}$$