Decomposition Theorems for Square-free 2-matchings in Bipartite Graphs

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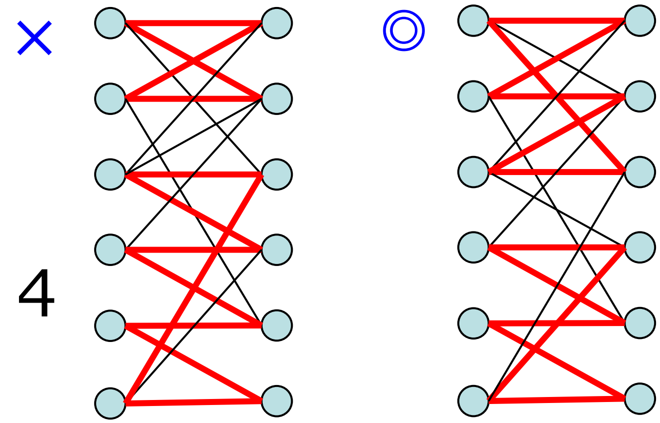
Overview

$G = (V,E)$: Bipartite, Simple

$M \subseteq E$: **Square-free 2-matching**

⇔ 2-matching w/o cycles of length 4

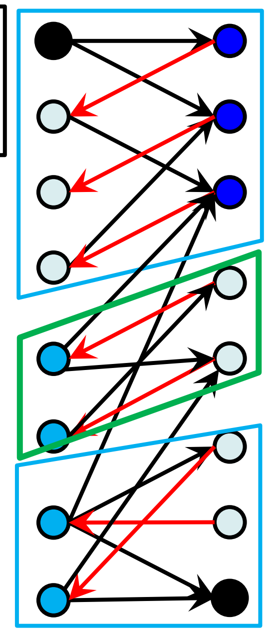


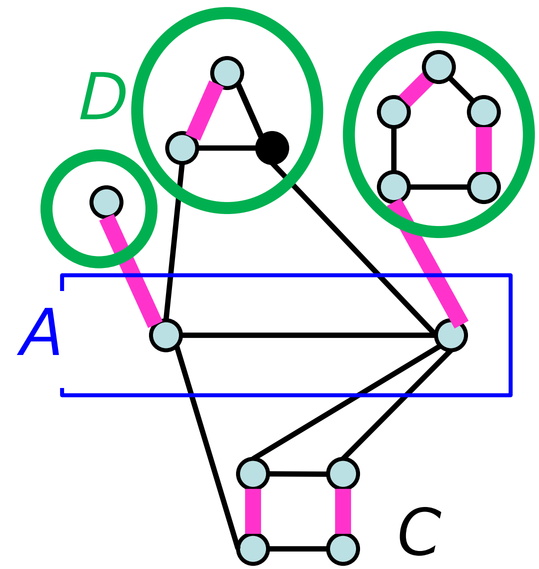
Dulmage-Mendelsohn decomposition for bipartite matching

↓

Edmonds-Gallai decomposition for nonbipartite matching

⇕ Similarity





[This talk] Decomposition of square-free 2-matchings in bipartite graphs

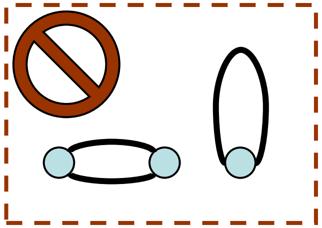
Contents

- C_k-free 2-matchings
  - Definition, Motivation
  - Previous work

- Classical decomposition theorems
  - DM-decomposition (Bipartite matching)
  - Edmonds-Gallai decomposition (Nonbipartite matching)

- Our result: Square-free 2-matchings in bipartite graphs
  - Analysis of two min-max theorems
    - ✓ [Király '99]
    - ✓ [Frank '03]
  - New decomposition theorems

C_k-free 2-matchings



$G = (V,E)$: Simple, Undirected

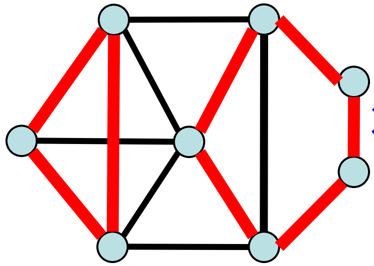
$k \in \mathbf{Z}_+$

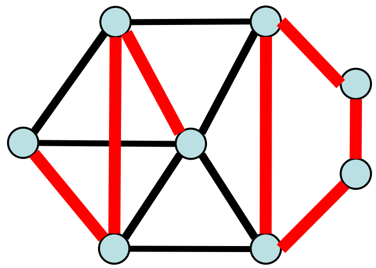
Def

Cycles + Paths

$M \subseteq E$ is a 2-matching
⇔ $\deg_M(v) \leq 2 \ \forall v \in V$

2-matching M is C_k-free
⇔ Free of cycles of length $\leq k$



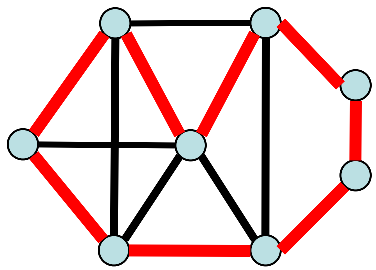


Problem

Find a C_k-free 2-matching M maximizing $|M|$

- $k \leq 2$ ➔ P
- $k \geq n/2$ ➔ NP-hard

(Opt $= n$ ➔ Hamilton cycle)



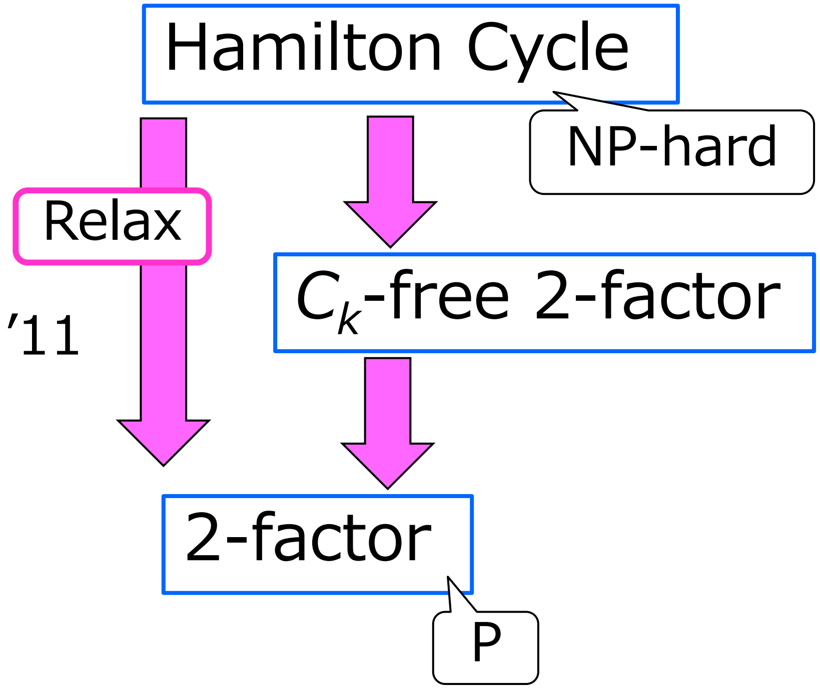
$n = |V|$

Motivation

◆ Relaxation of the Hamilton cycle problem

➢ Approaches to the TSP via C_k-free 2-factors

- Boyd, Iwata, T. '11
- Boyd, Sitters, van der Ster, Stougie '11
- Correa, Larré, Soto '12
- Karp, Ravi '14
- T. '15



◆ Connectivity augmentation

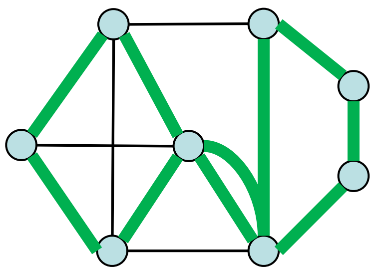
G = (V,E) is $(n-3)$-vertex connected
⟺ Complement of G is a 2-matching without C_4

- Bérczi, Kobayashi '12

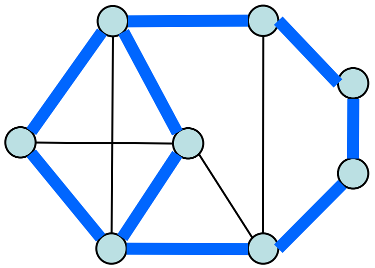
Application to Approximation

◆ Relaxation of the Hamilton cycle problem

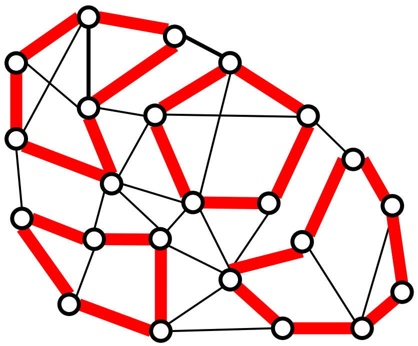
➢ Graph-TSP
(= Min. spanning Eulerian subgraph)



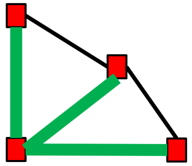
➢ Min. 2-edge conn. subgraph



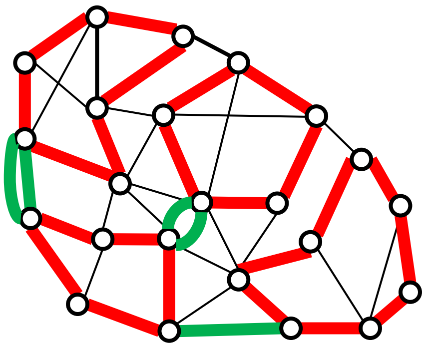
C_k-free 2-factor



Spanning tree
$\leq \frac{n}{k+1} - 1$ edges



Solution $\leq n + \frac{2n}{k+1} - 2$ edges

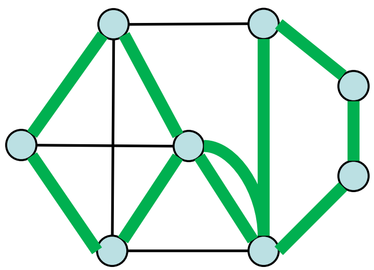


C_k-free 2-factor ➔ $(1 + \frac{2}{k+1})$-approximation

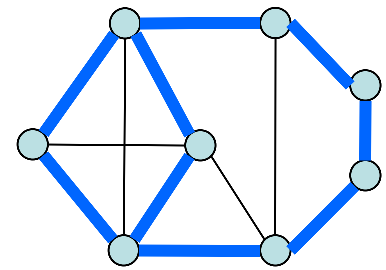
Application to Approximation

◆ Relaxation of the Hamilton cycle problem

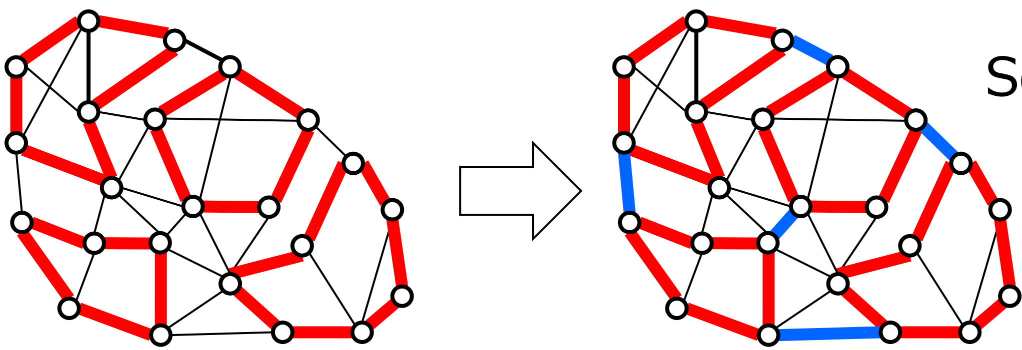
- Graph-TSP (= Min. spanning Eulerian subgraph)



- Min. 2-edge conn. subgraph



C_k-free 2-factor



Solution $\leq n + \frac{2n}{k+1} - 2$ edges

C_k-free 2-factor ➔ $(1 + \frac{2}{k+1})$-approximation

Complexity

General	Unweighted	Weighted
$k \geq n/2$	NP-hard	NP-hard
$k \geq 5$	NP-hard (Papadimitriou '78)	NP-hard
$k = 4$	OPEN	NP-hard (Vornberger '80)
$k = 3$	P (Hartvigsen '84)	OPEN
$k = 2$	P	P

Bipartite	Unweighted	Weighted
$k \geq n/2$	NP-hard	NP-hard
$k \geq 6$	NP-hard (Geelen '99)	NP-hard (Geelen '99)
$k = 4$	P (Hartvigsen '06, Pap '07)	NP-hard (Király '03)
$k = 2$	P	P

Edge-weight is vertex-induced on every square:

- Polyhedral description with dual integrality [Makai '07]
- Combinatorial algorithm [T. '09]
- M-convex fn. on a jump system [Kobayashi, Szabó, T. '12]

Square-free, Bipartite: Well-solved

Square-free 2-matchings in bipartite graphs

- Min-max theorems [Király '99] [Frank '03]
- Algorithms [Hartvigsen '06] [Pap '07] [T. '09] [Babenko '12]
- Dual integrality [Makai '07]
- Discrete convexity [Szabó, Kobayashi, T. '12]
- **Decomposition Theorem [This talk]**

Matchings

- Min-max theorem [Tutte '47] [Berge '58]
- Algorithms [Edmonds '65] etc.
- Total dual integrality [Edmonds '65] [Cunningham-Marsh '78]
- Discrete convexity [Chandrasekaran-Kabadi '88] [Bouchet '89] [Murota '97]
- Decomposition theorem [Dulmage-Mendelsohn '58] [Gallai '63] [Edmonds '65]

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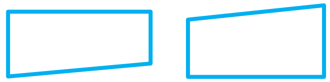
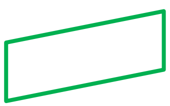
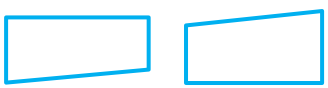
    - [Király '99]
    - [Frank '03]
  - New decomposition theorems

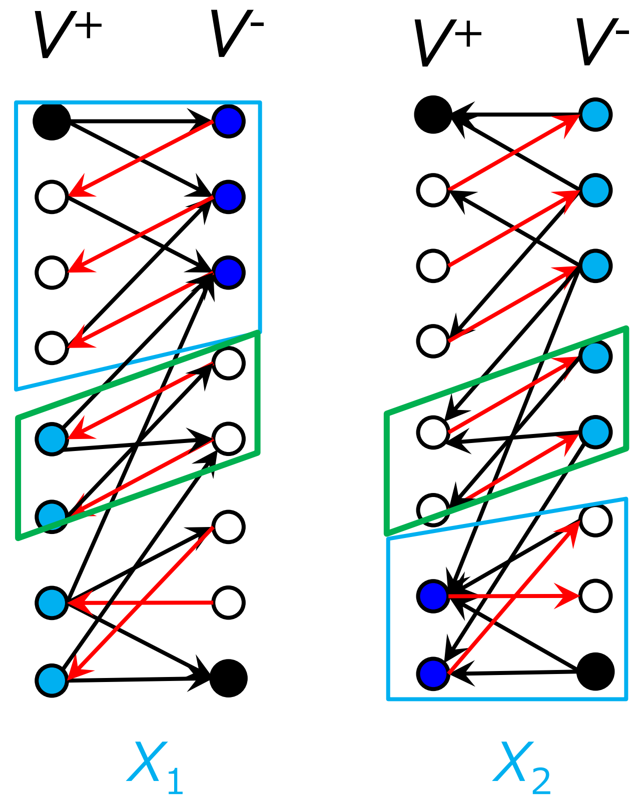
DM-decomposition

Theorem [Kőnig '31]

$\max\{|M| : M$ is a matching$\}$
$= \min\{|X| : X \subseteq V, X$ is a vertex cover$\}$

Theorem [DM-decomposition]

- $Y \subseteq V$: min. vertex cover
  $\Rightarrow X_2^+ \subseteq Y^+ \subseteq X_1^+,\ X_1^- \subseteq Y^- \subseteq X_2^-$
- Each edge in 
  is contained in some max. matching
-  has a perfect matching
- $M \subseteq E$: max. matching
  $\Leftrightarrow M$ consists of :
  - Max. matchings in 
  - Perfect matching in 



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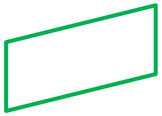
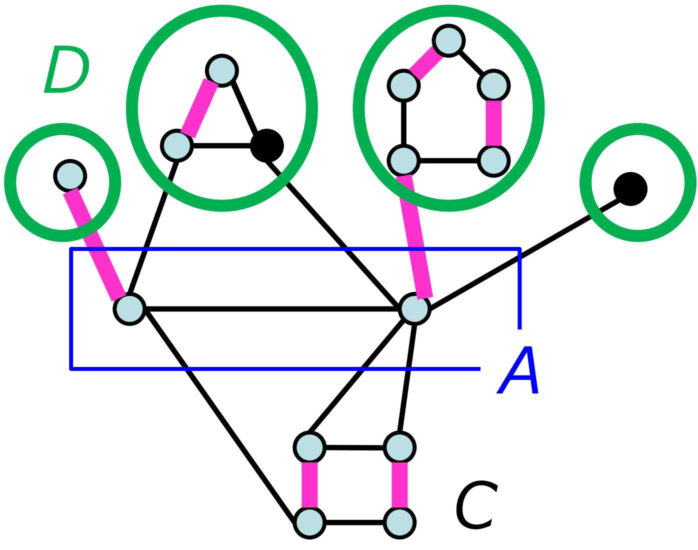
- Lattice structure in 

Characterize max matching & min cover

Edmonds-Gallai decomposition

Theorem [Tutte '47, Berge '58]

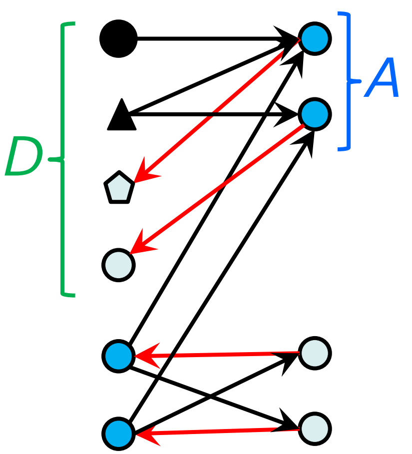
max{|M| : M is a matching}
$= \frac{1}{2}\min\{|V| + |X| - \mathrm{odd}(\bar{X}) : X \subseteq V\}$



D = {$v \in V$: ∃max. matching missing v}
A = {$v \in V$ - D : v is adjacent to D}
C = V – D - A

Theorem [Edmonds-Gallai decomposition]

- max{|M|} = $(|V| + |A| - \mathrm{odd}(\bar{A}))/2$
- Components in $G[D]$ are factor-critical
- $G[C]$ has a perfect matching
- M: max. matching in G ➔
  - $M[D]$: near-perfect matching in each component
  - $M[D,A]$ matches vertices in A with distinct components in D
  - $M[C]$: perfect matching in $G[C]$
- Delete C, $E[A]$, contract the components in $G[D]$
  ➔ A bipartite graph with $\Gamma(X) > |X|$ $\forall X \subseteq A$



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    - [Király '99]
    - [Frank '03]
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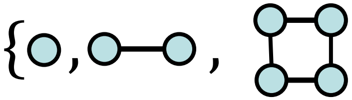
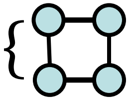
Two min-max theorems

$G = (V,E)$: Bipartite

$q(\bar{X}) = \#\{$  in $G[\bar{X}]\}$

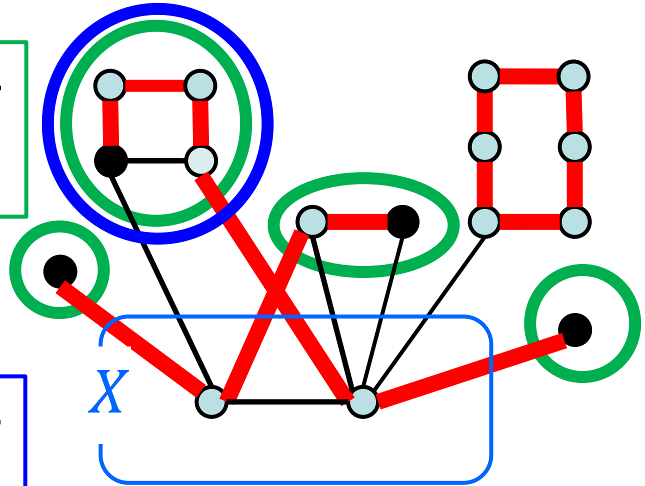
$c(\bar{X}) = \#\{$  in $G[\bar{X}]\}$

Theorem [Király '99]

max $\{|M|$: M is a sq.-free 2-matching$\}$
$= \min\{|V| + |X| - q(\bar{X}) : X \subseteq V\}$

Theorem [Frank '03]

max $\{|M|$: M is a sq.-free 2-matching$\}$
$= \min\{2|X| + |E[\bar{X}]| - c(\bar{X}) : X \subseteq V\}$



$q(\bar{X}) = 4$

$c(\bar{X}) = 1$

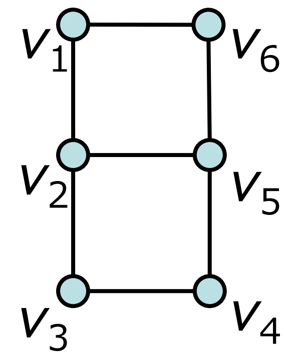
Nonbipartite matching

max $\{|M|$: M is a matching$\}$
$= \frac{1}{2}\min\{|V| + |X| - \mathrm{odd}(\bar{X}) : X \subseteq V\}$

Bipartite 2-matching

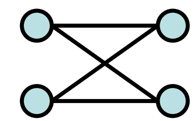
max$\{|M|$: M is a 2-matching$\}$
$= \min\{2|X| + |E[\bar{X}]| : X \subseteq V\}$

Király vs. Frank

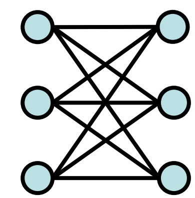


- [Király '99] ≅ Nonbipartite matching
  [Frank '03] ≅ Bipartite 2-matching
- $X_1 = \{v_1,v_2,v_3,v_4,v_5,v_6\}$: Minimizing [Király '99]
  Not minimizing [Frank '03]

  $X_2 = \{v_1,v_2,v_3,v_4,v_6\}$: Minimizing [Frank '03]
  Not minimizing [Király '99]



$K_{2,2}$



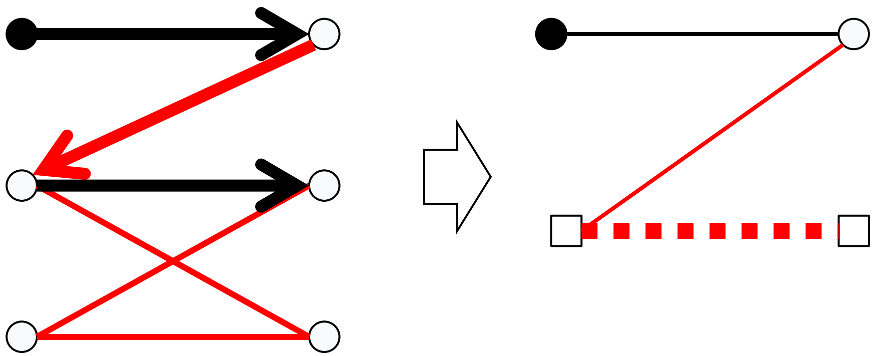
$K_{3,3}$

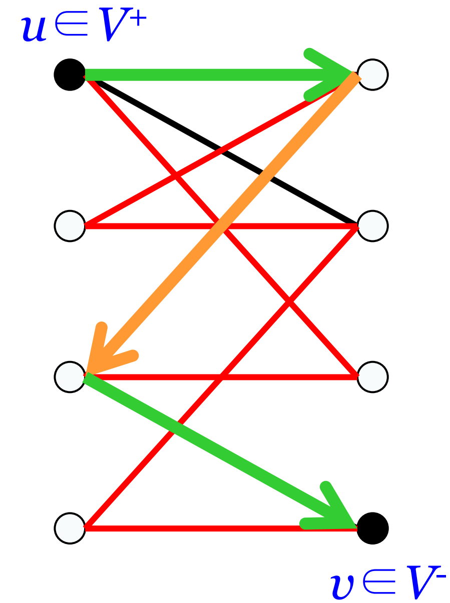
- [Frank '03] extends to $K_{t,t}$-free t-matchings
- [Király '99] applies to decomposition theorems
- Algorithms
  - [Hartvigsen '06] : Minimizing both [Király '99], [Frank '03]
  - [Pap '07] : Minimizing [Frank '03]

$\{v_1,v_3,v_5\}$

Algorithm [Hartvigsen '06]

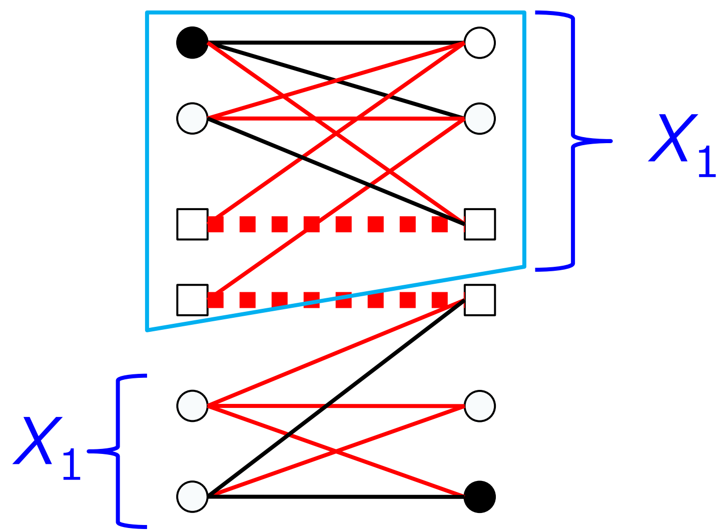
- Search an augmenting path P ➔ $M' := M \Delta P$
- A square contained in M'
  ➔ Contract the square





- No augmenting path ➔
  - ✓ Max. square-free 2-matching M
  - ✓ Minimizer X_1

Both in [Király '99], [Frank '03]



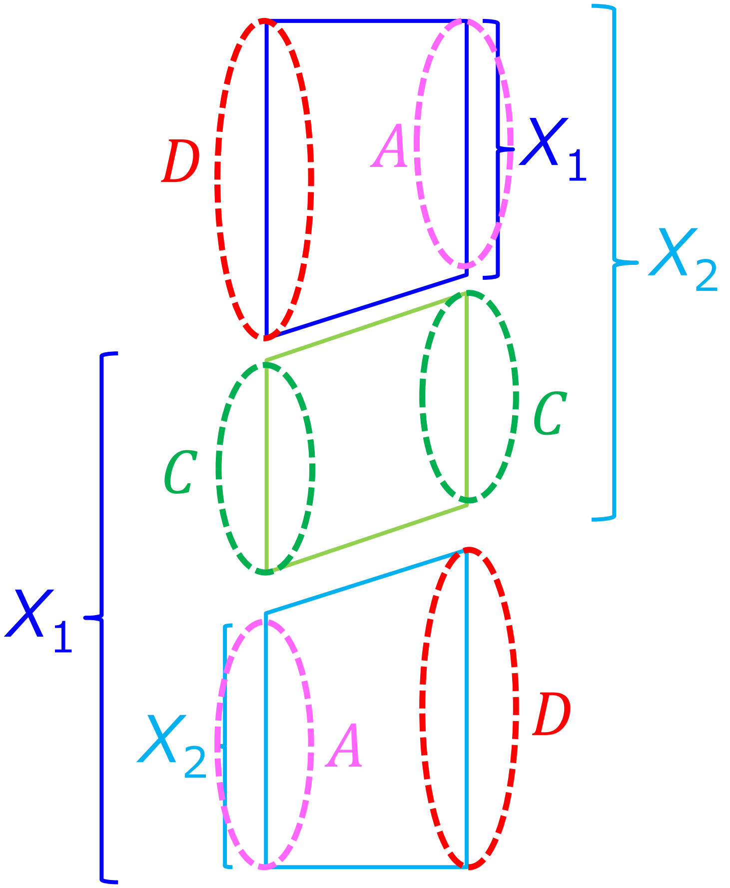
Dulmage-Mendelsohn Structure

X_1 : Minimizer (V^+ : source)
X_2 : Minimizer (V^- : source)

Theorem

$Y \subseteq V$ minimizes [Király '99]
$\Rightarrow X_2^+ \subseteq Y^+ \subseteq X_1^+, X_1^- \subseteq Y^- \subseteq X_2^-$

$D = (V^+ - X_1) \cup (V^- - X_2)$
$A = (V^+ \cap X_2) \cup (V^- \cap X_1)$
$C = V - D - A$



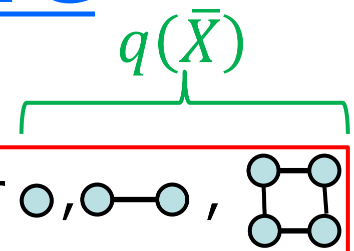
Theorem

$D = \{u \in V : {}^{\exists}$max. square-free 2-matching M, $\deg_M(u) < 2\}$

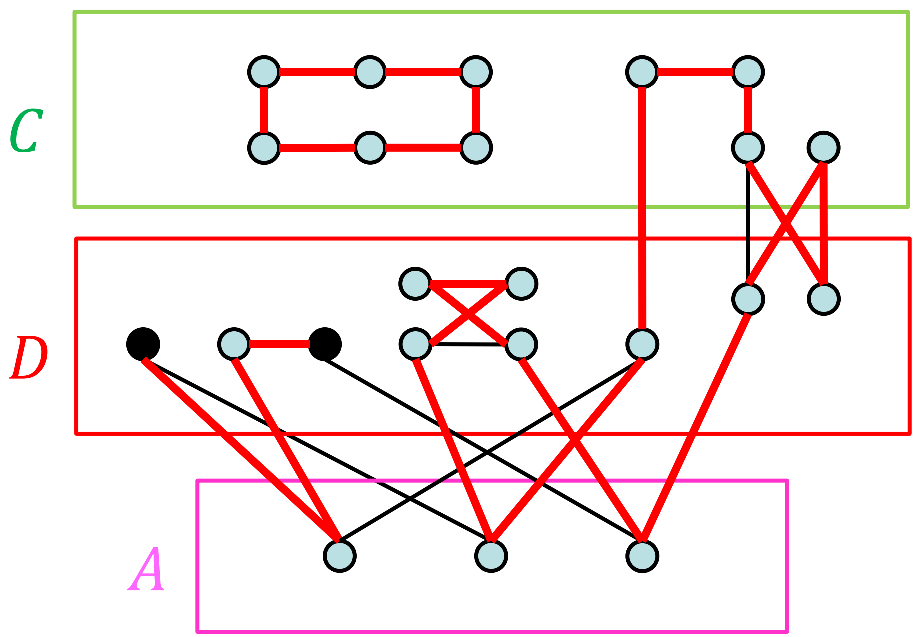
Each edge in $E[D,A]$ is contained in some max. square-free 2-matching

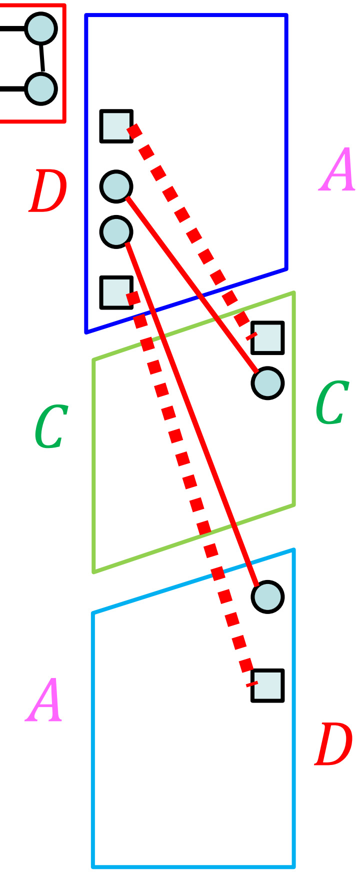
Edmonds-Gallai Structure

Theorem

- Components in $G[D]$ and $G[D,C]$ are either 

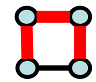
- Contract the squares in $G[D]$, $G[D,C]$;
  For $u \in C$, let $b(u) = 1$ if u is adjacent to D,
  $b(u) = 2$ o.w.
  - $G[C]$ has a b-factor
  - $b(\Gamma(X) \cap D) > 2|X|$ $\forall X \subseteq A$



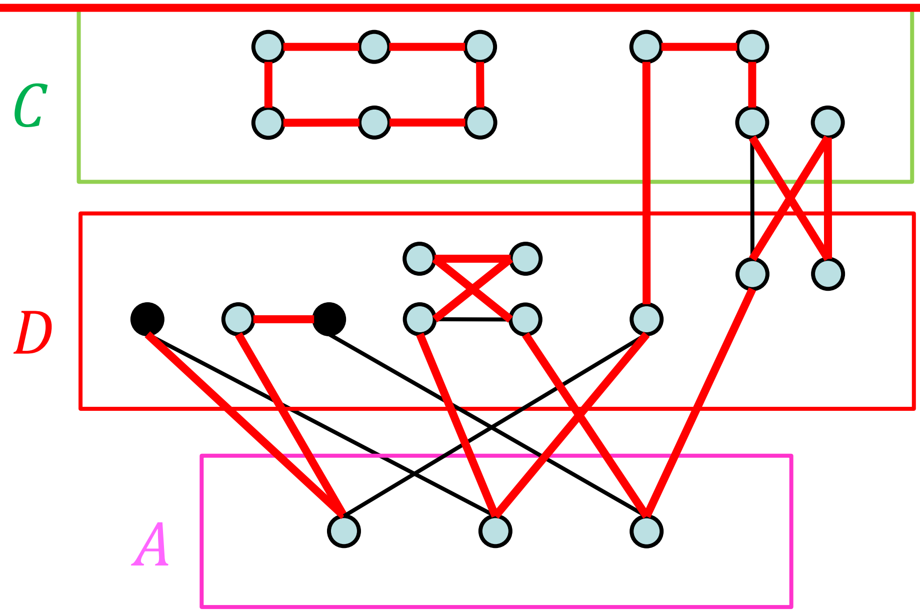


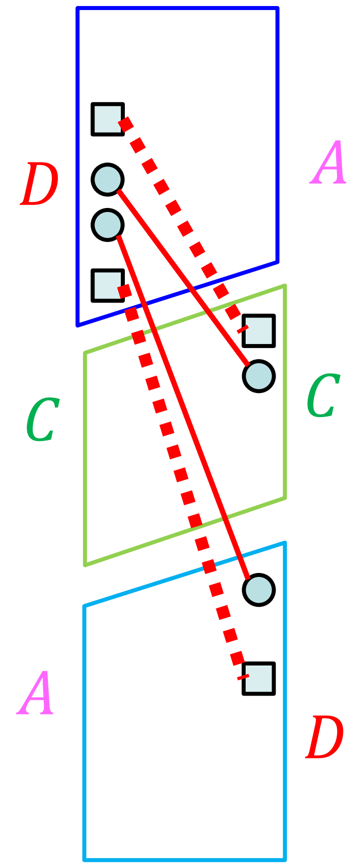
Edmonds-Gallai Structure

Theorem

- M: max. square-free 2-matching ➔
  - $M[D]$, $M[D,C]$: contain 1 edge in 
    3 edges in 
  - $M[C]$: b-factor in $G[C]$
    i.e., $\deg_M(u)=2\ ^{\forall}u\in C$
  - $M[D,A]$: matches each vertex in A
    to distinct components in D

Characterize max. square-free 2-matchings





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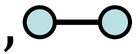
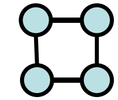
Summary & Future work

- Square-free 2-matchings in bipartite graphs: Well-solved
  - Min-max theorems
  - Polyhedral description with dual integrality
  - Algorithms
  - Discrete convexity
  - **Decomposition theorems [This talk]**

- Lattice structure ??
- Algorithmic application
  cf. [Karp, Ravi '14]
  9/7-approx. for the graph-TSP in cubic bipartite graphs
- Open problems:
  - C_4-free 2-matching — Discrete convexity [Kobayashi, Szabó, T. '12]
  - Weighted C_3-free 2-matching — [Kobayashi '14]
  - Barnette conjecture: 3-conn. planar cubic bipartite $\overset{??}{\Rightarrow}$ Hamiltonian

Min-max theorem [Király '99]

$G = (V,E)$: Bipartite

$q(\bar{X}) = \#\{$  ,  ,  in $G[\bar{X}]\}$

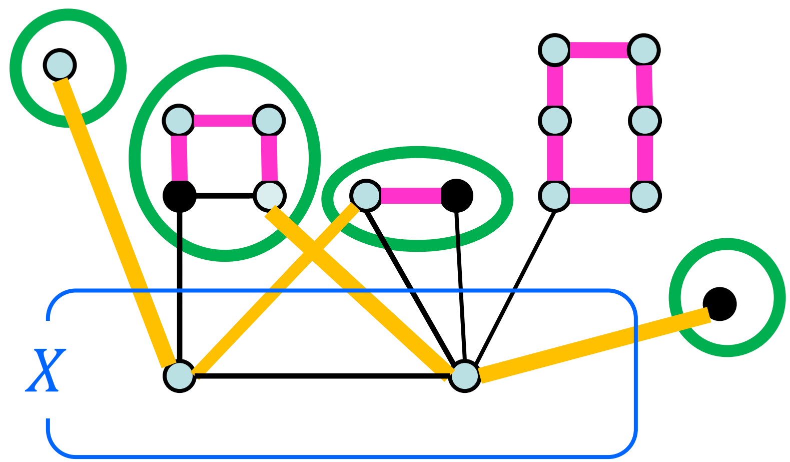
Thm [Király '99]

max $\{|M|$: M is a C_4-free 2-matching$\}$

$= \min\{|V| + |X| - q(\bar{X}) : X \subseteq V\}$

[max ≤ min]

- $|M| = |M[\bar{X}]| + |M[X,\bar{X}]| + |M[X]|$
- $|M[X,\bar{X}]| + 2|M[X]| \leq 2|X|$
- $|M[\bar{X}]| \leq |\bar{X}| - q(\bar{X})$



$q(\bar{X}) = 4$

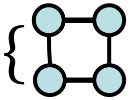
Tutte-Berge formula (Nonbipartite matching)

max $\{|M|$: M is a matching$\}$

$= \frac{1}{2}\min\{|V| + |X| - \mathrm{odd}(\bar{X}) : X \subseteq V\}$

Min-max theorem [Frank '03]

$G = (V,E)$: Bipartite

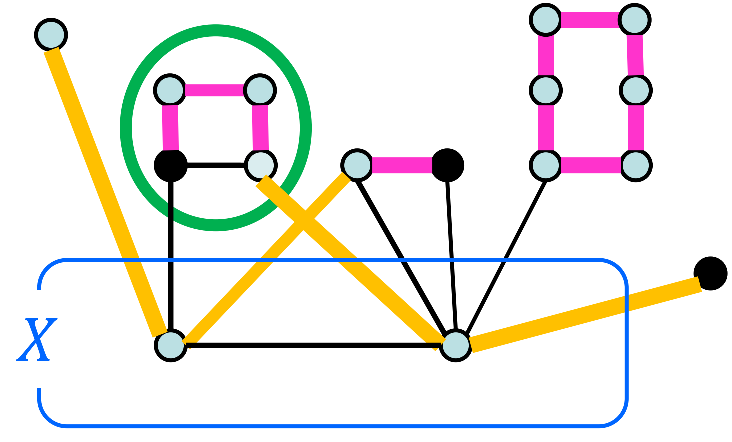
$c(\bar{X}) = \#\{$  in $G[\bar{X}]\}$

Thm [Frank '03]

max $\{|M|$: M is a C_4-free 2-matching$\}$
$= \min\{2|X| + E[\bar{X}] - c(\bar{X}) : X \subseteq V\}$

[max ≤ min]

- $|M| = |M[\bar{X}]| + |M[X,\bar{X}]| + |M[X]|$
- $|M[X,\bar{X}]| + 2|M[X]| \leq 2|X|$
- $|M[\bar{X}]| \leq |E[\bar{X}]| - c(\bar{X})$



$c(\bar{X}) = 1$

Thm (Bipartite 2-matching)

max$\{|M|$: M is a 2-matching$\}$
$= \min\{2|X| + |E[\bar{X}]| : X \subseteq V\}$