

Structure Theorems for Square-free 2-matchings in Bipartite Graphs

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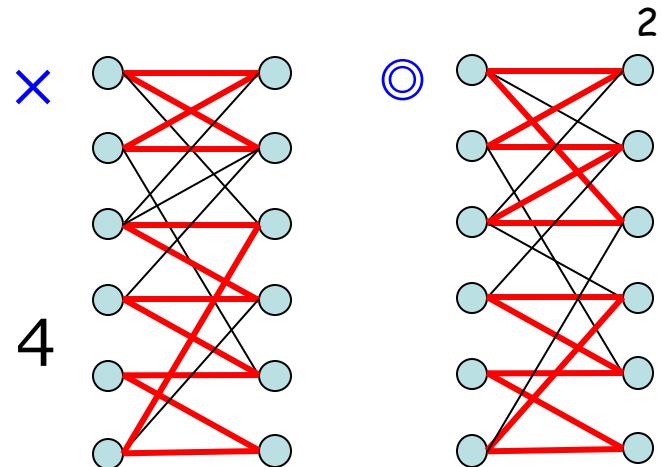
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Overview

$G = (V, E)$: Bipartite, Simple

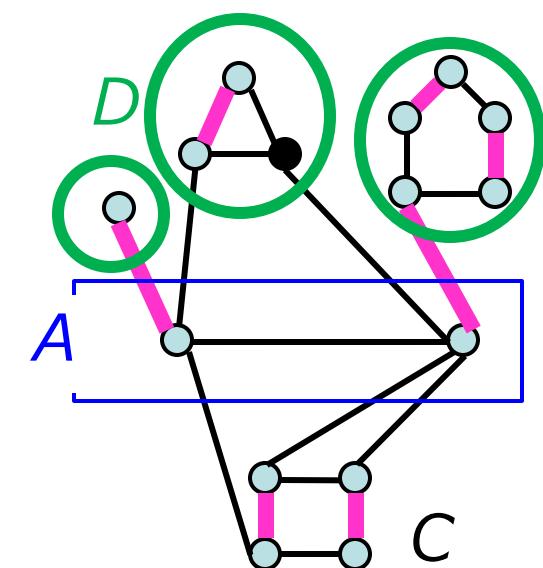
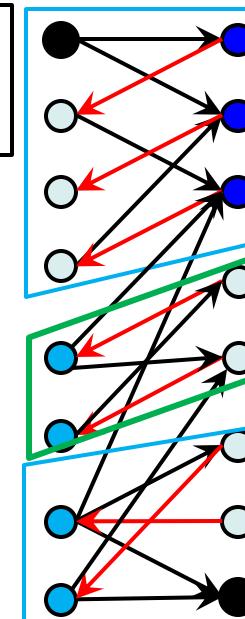
$M \subseteq E$: **Square-free 2-matching**

↔ 2-matching w/o cycles of length 4



Dulmage-Mendelsohn decomposition
for bipartite matching

Edmonds-Gallai decomposition
for nonbipartite matching



[This talk] Decomposition of square-free 2-matchings
in bipartite graphs

Contents

◆ C_k -free 2-matchings

- Definition, Motivation
- Previous work

Bérczi, Frank, Z. Király, Kobayashi,
Makai, Gy. Pap, Szabó, Végh ...

◆ Classical decomposition theorems

- DM-decomposition (Bipartite matching)
- Edmonds-Gallai decomposition (Nonbipartite matching)

◆ Our result: Square-free 2-matchings in bipartite graphs

- Analysis of two min-max theorems
 - ✓ [Király '99]
 - ✓ [Frank '03]
- New decomposition theorems

C_k -free 2-matchings

$G = (V, E)$: Simple, Undirected

$k \in \mathbb{Z}_+$

Def

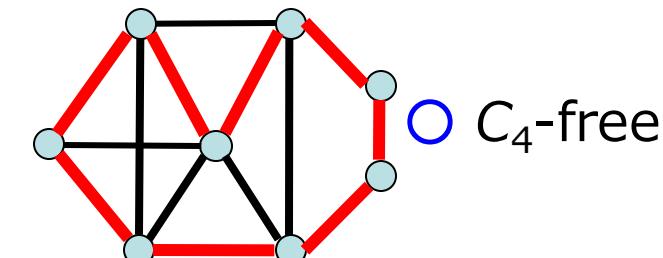
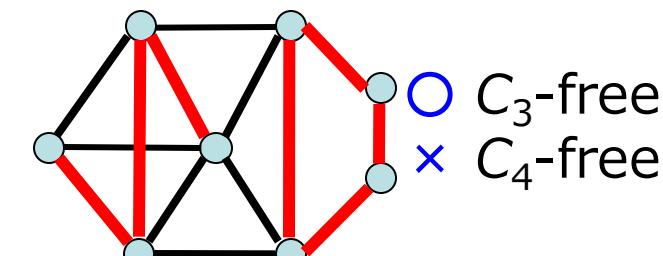
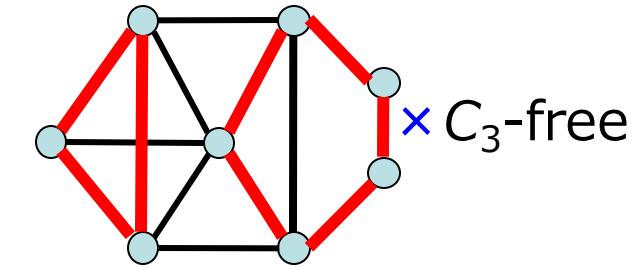
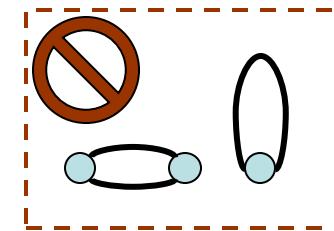
$M \subseteq E$ is a **2-matching**

$$\iff \deg_M(v) \leq 2 \quad \forall v \in V$$

Cycles + Paths

2-matching M is C_k -free

\iff Free of cycles of length $\leq k$



Problem

Find a C_k -free 2-matching M
maximizing $|M|$

$\triangleright k \leq 2 \rightarrow P$

$\triangleright k \geq n/2 \rightarrow \text{NP-hard}$

(Opt = $n \rightarrow$ Hamilton cycle)

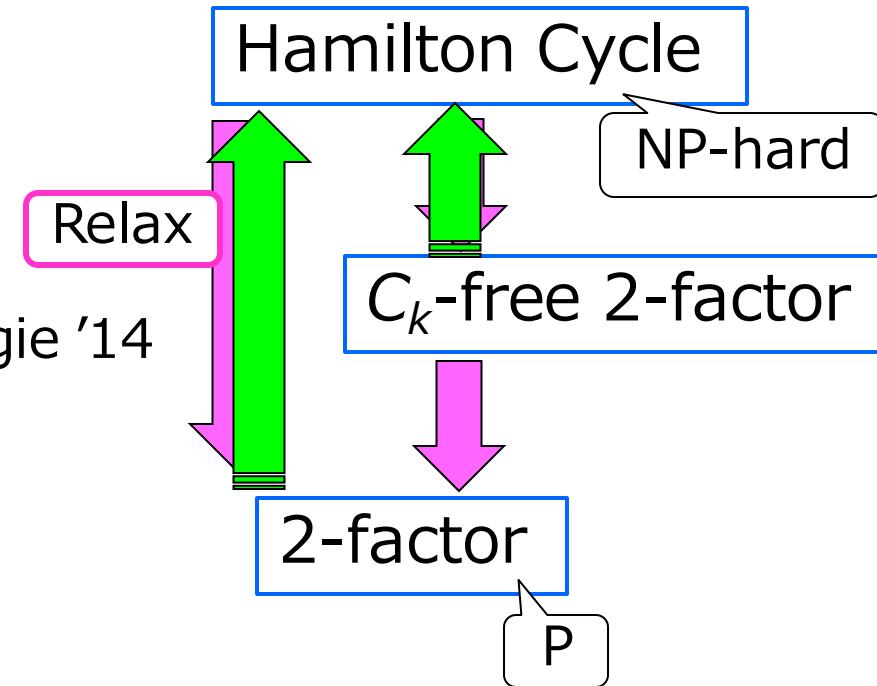
$n = |V|$

Motivation

◆ Relaxation of the Hamilton cycle problem

➤ Approaches to the TSP via C_k -free 2-factors

- Boyd, Iwata, T. '13
- Boyd, Sitters, van der Ster, Stougie '14
- Correa, Larré, Soto '12
- Karp, Ravi '14



◆ Connectivity augmentation

$G = (V, E)$ is $(n - 3)$ -vertex connected

↔ Complement of G is a 2-matching without C_4

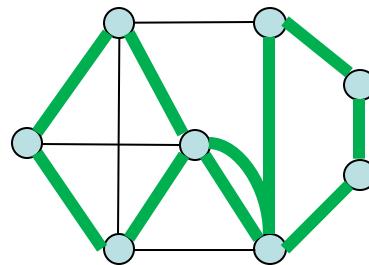
- Bérczi, Kobayashi '12

Application to Approximation

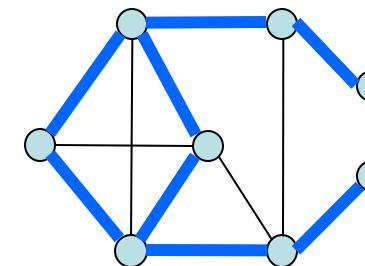
◆ Relaxation of the Hamilton cycle problem

➤ Graph-TSP

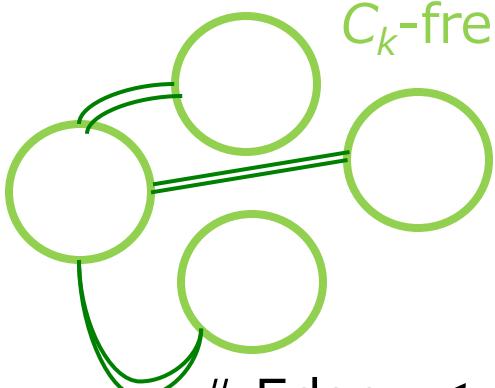
(= Min. spanning Eulerian subgraph)



➤ Min. 2-edge conn. subgraph

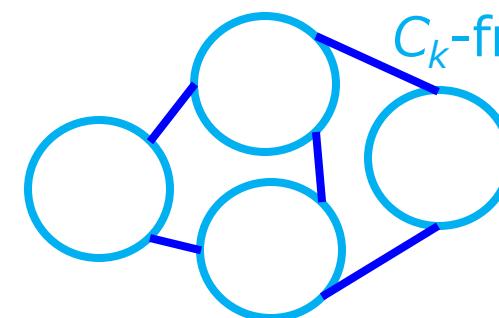


C_k -free 2-factor



$$\# \text{ Edges} \leq n + 2 \cdot \left(\frac{n}{k+1} - 1 \right)$$

C_k -free 2-factor



$$\# \text{ Edges} \leq n + 2 \cdot \left(\frac{n}{k+1} - 1 \right)$$

C_k -free 2-factor $\rightarrow (1 + \frac{2}{k+1})$ -approximation

Complexity

General	Unweighted	Weighted
$k \geq n/2$	NP-hard	NP-hard
$k \geq 5$	NP-hard (Papadimitriou '78)	NP-hard
$k = 4$	OPEN	NP-hard (Vornberger '80)
$k = 3$	P (Hartvigsen '84)	OPEN
$k = 2$	P	P

Bipartite	Unweighted	Weighted
$k \geq n/2$	NP-hard	NP-hard
$k \geq 6$	NP-hard (Geelen '99)	NP-hard (Geelen '99)
$k = 4$	P (Hartvigsen '06, Pap '07)	NP-hard (Király '03)
$k = 2$	P	P

Edge-weight is **vertex-induced** on every square:

- Subc ➤ Polyhedral description with dual integrality [Makai '07]
 ➤ Combinatorial algorithm [T. '09]
 ➤ M-convex fn. on a jump system [Kobayashi, Szabó, T. '12]

Square-free, Bipartite: Well-solved

Square-free 2-matchings in bipartite graphs

- Min-max theorems [Király '99] [Frank '03]
- Algorithms [Hartvigsen '06] [Pap '07] [T. '09] [Babenko '12]
- Dual integrality [Makai '07]
- Discrete convexity [Szabó, Kobayashi, T. '12]

Matchings

- Min-max theorem [Tutte '47] [Berge '58]
- Algorithms [Edmonds '65] etc.
- Total dual integrality [Edmonds '65] [Cunningham-Marsh '78]
- Discrete convexity [Chandrasekaran-Kabadi '88] [Bouchet '89]
[Murota '97]
- Decomposition [Dulmage-Mendelsohn '58]
[Gallai '63] [Edmonds '65]

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◆ Our result: Square-free 2-matchings in bipartite graphs

- Analysis of two min-max theorems
 - ✓ [Király '99]
 - ✓ [Frank '03]
- New decomposition theorems

DM-decomposition

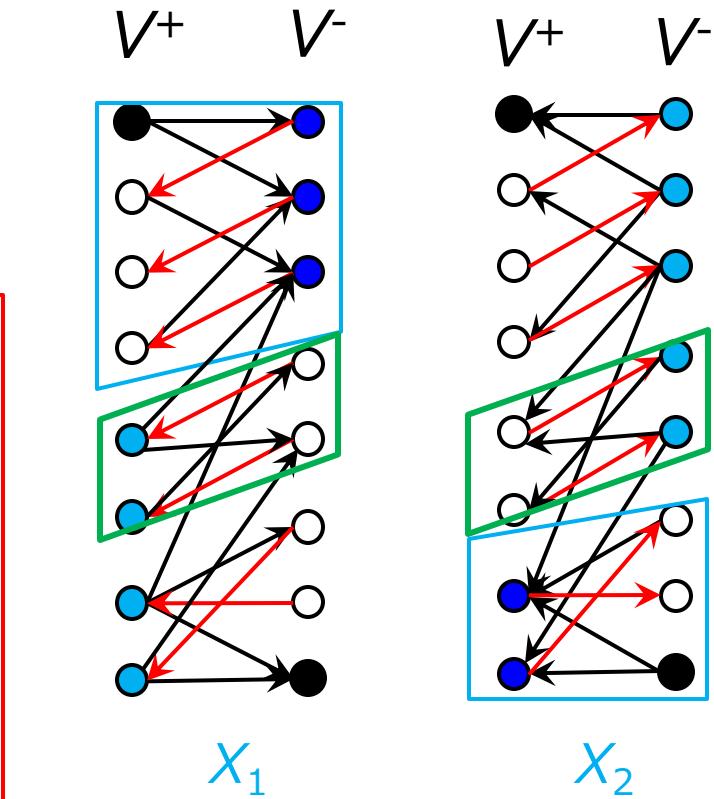
Theorem [König '31]

$$\max\{|M| : M \text{ is a matching}\} = \min\{|X| : X \subseteq V, X \text{ is a vertex cover}\}$$

Theorem [DM-decomposition]

- $Y \subseteq V$: min. vertex cover
 $\Rightarrow X_2^+ \subseteq Y^+ \subseteq X_1^+, X_1^- \subseteq Y^- \subseteq X_2^-$
- Each edge in   is contained in some max. matching
-  has a perfect matching
- $M \subseteq E$: max. matching
 $\Leftrightarrow M$ consists of :
 - Max. matchings in  
 - Perfect matching in 

Characterize max matching & min cover



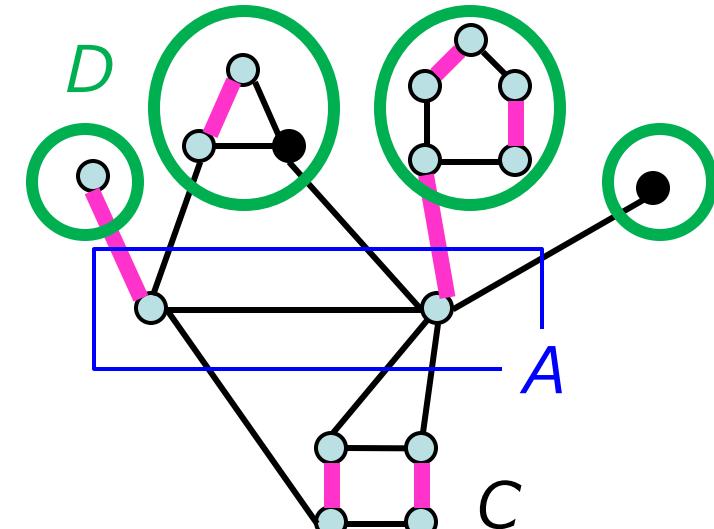
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- Lattice structure in 

Edmonds-Gallai decomposition

Theorem [Tutte '47, 'Berge '58]

$$\begin{aligned} \max\{|M| : M \text{ is a matching}\} \\ = \frac{1}{2} \min\{|V| + |X| - \text{odd}(\bar{X}) : X \subseteq V\} \end{aligned}$$



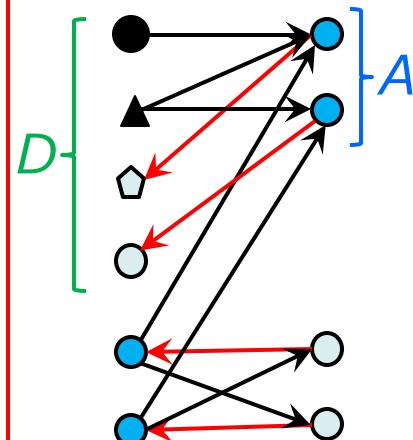
$$D = \{v \in V : \exists \text{ max. matching missing } v\}$$

$$A = \{v \in V - D : v \text{ is adjacent to } D\}$$

$$C = V - D - A$$

Theorem [Edmonds-Gallai decomposition]

- $\max\{|M|\} = (|V| + |A| - \text{odd}(\bar{A})) / 2$
- Components in $G[D]$ are factor-critical
- $G[C]$ has a perfect matching
- M : max. matching in $G \rightarrow$
 - $M[D]$: near-perfect matching in each component
 - $M[D, A]$ matches vertices in A with distinct components in D
 - $M[C]$: perfect matching in $G[C]$
- Delete C , $E[A]$, contract the components in $G[D]$
→ A bipartite graph with $\Gamma(X) > |X| \forall X \subseteq A$



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 - ✓ [Frank '03]
- New decomposition theorems

Two min-max theorems

$G = (V, E)$: Bipartite

$$q(\bar{X}) = \#\{\textcircled{o}, \textcircled{o} - \textcircled{o}, \textcircled{o} - \textcircled{o} - \textcircled{o} \text{ in } G[\bar{X}]\}$$

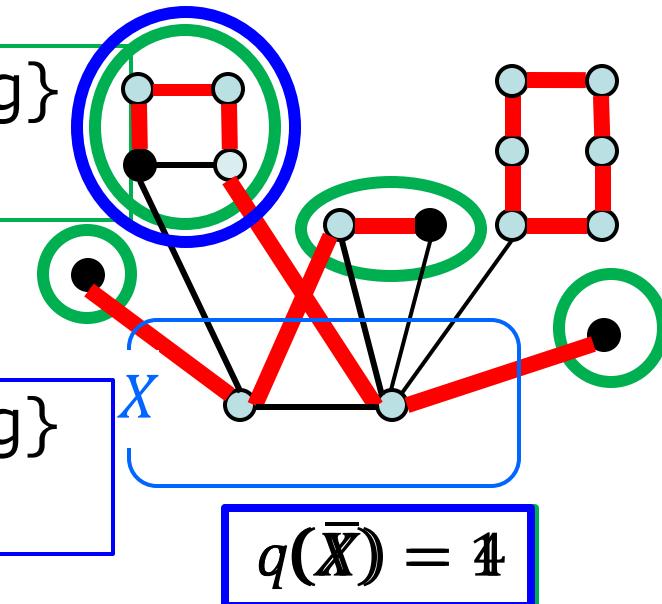
$$c(\bar{X}) = \#\{\textcircled{o} - \textcircled{o} - \textcircled{o} - \textcircled{o} \text{ in } G[\bar{X}]\}$$

Theorem [Király '99]

$$\begin{aligned} & \max \{|M| : M \text{ is a sq.-free 2-matching}\} \\ &= \min\{|V| + |X| - q(\bar{X}) : X \subseteq V\} \end{aligned}$$

Theorem [Frank '03]

$$\begin{aligned} & \max \{|M| : M \text{ is a sq.-free 2-matching}\} \\ &= \min\{2|X| + |E[\bar{X}]| - c(\bar{X}) : X \subseteq V\} \end{aligned}$$



Nonbipartite matching

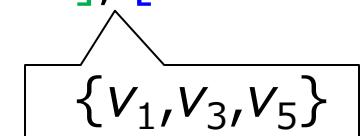
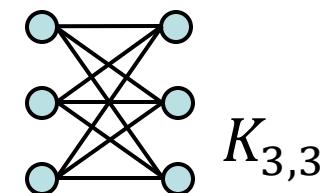
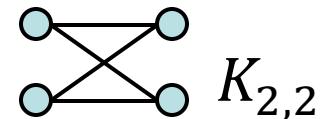
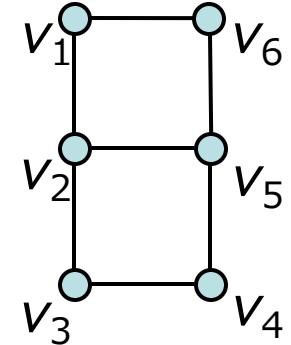
$$\begin{aligned} & \max \{|M| : M \text{ is a matching}\} \\ &= \frac{1}{2} \min \{|V| + |X| - \text{odd}(\bar{X}) : X \subseteq V\} \end{aligned}$$

Bipartite 2-matching

$$\begin{aligned} & \max \{|M| : M \text{ is a 2-matching}\} \\ &= \min\{2|X| + |E[\bar{X}]| : X \subseteq V\} \end{aligned}$$

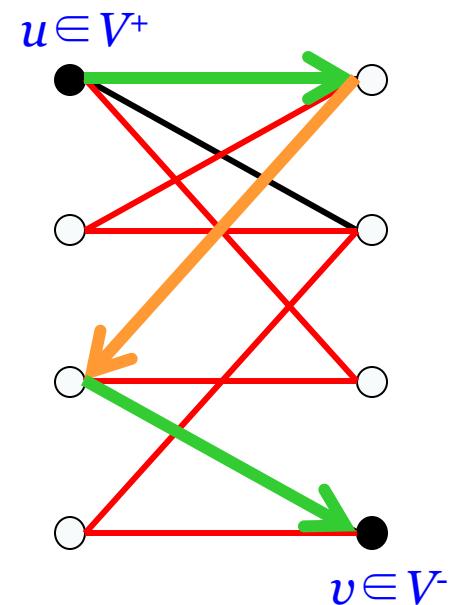
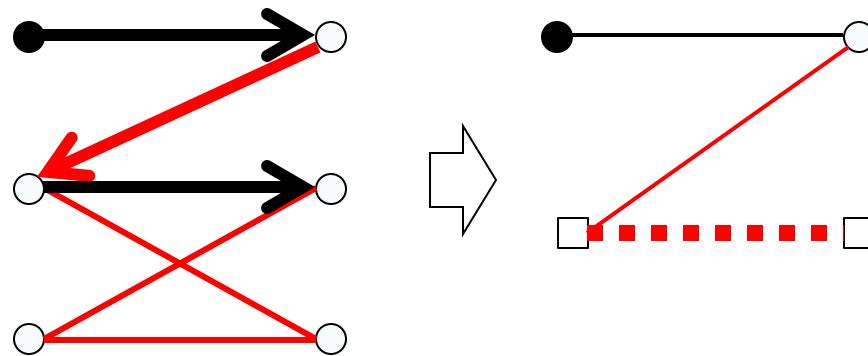
Király vs. Frank

- [Király '99] \cong Nonbipartite matching
 [Frank '03] \cong Bipartite 2-matching
- $X_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$: Minimizing [Király '99]
 Not minimizing [Frank '03]
- $X_2 = \{v_1, v_2, v_3, v_4, v_6\}$: Minimizing [Frank '03]
 Not minimizing [Király '99]
- [Frank '03] extends to $K_{t,t}$ -free t -matchings
- [Király '99] applies to decomposition theorems
- Algorithms
 - [Hartvigsen '06] : Minimizing both [Király '99], [Frank '03]
 - [Pap '07] : Minimizing [Frank '03]



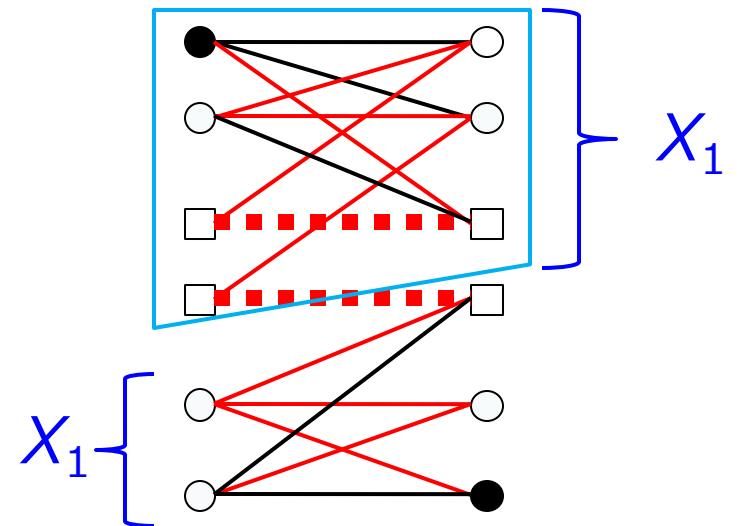
Algorithm [Hartvigsen '06]

- Search an augmenting path $P \rightarrow M' := M \Delta P$
- If a square contained in M' is found
→ Contract the square



- No augmenting path →
 - ✓ Max. square-free 2-matching M
 - ✓ Minimizer X_1

Both in [Király '99], [Frank '03]



Canonical minimizers

X_1 : Minimizer (V^+ : source)

X_2 : Minimizer (V^- : source)

Theorem

$Y \subseteq V$ minimizes [Király '99]

$\Rightarrow X_2^+ \subseteq Y^+ \subseteq X_1^+, X_1^- \subseteq Y^- \subseteq X_2^-$

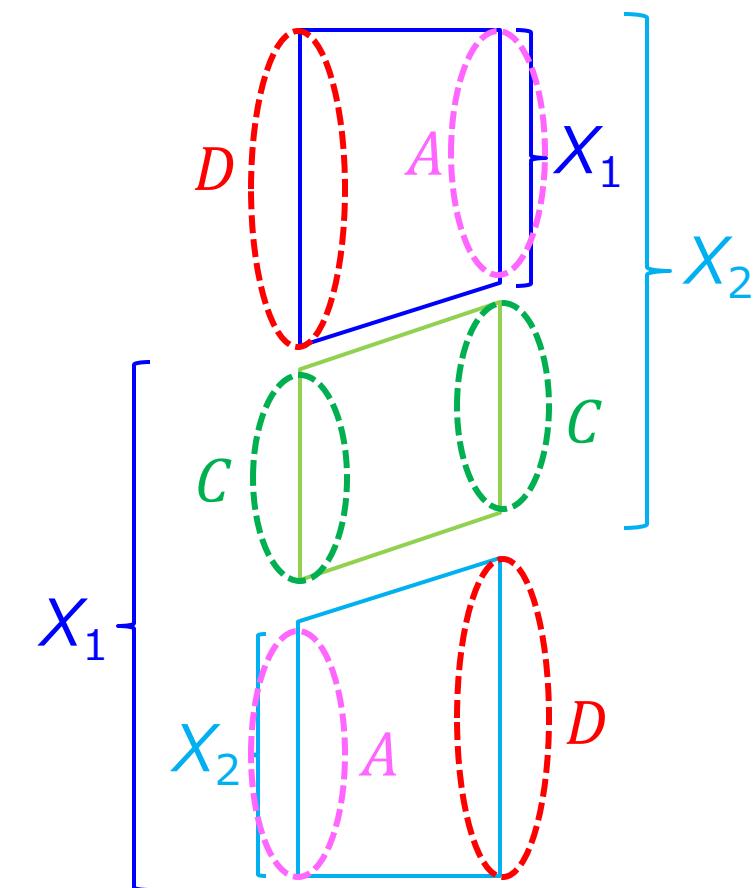
$$D = (V^+ - X_1) \cup (V^- - X_2)$$

$$A = (V^+ \cap X_2) \cup (V^- \cap X_1)$$

$$C = V - D - A$$

Theorem

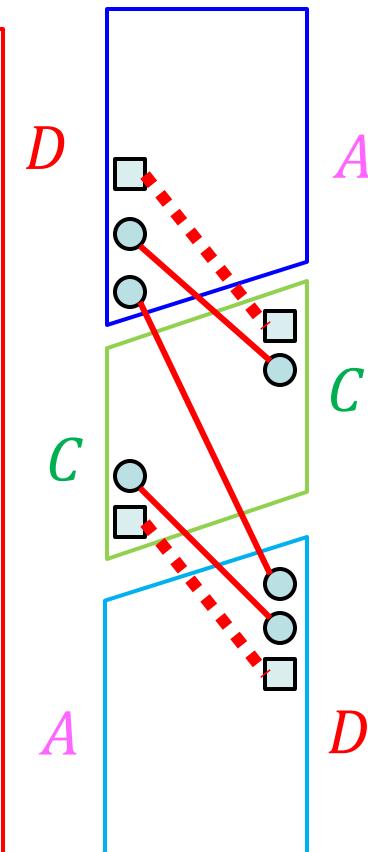
$D = \{u \in V : \exists \text{max. square-free 2-matching } M, \deg_M(u) < 2\}$



Decomposition theorem

Theorem

- Each edge in $E[D, A]$ is contained in some max. square-free 2-matching
- Components in $G[D]$, $G[D, C]$ are either  ,  , 
- Contract the squares in $G[D]$, $G[D, C]$;
For $u \in C$, let $b(u) = 1$ if u is adjacent to D ,
 $b(u) = 2$ otherwise
 - $G[C]$ has a b -factor
 - $b(\Gamma(X) \cap D) > 2|X| \quad \forall X \subseteq A$
- M : max. square-free 2-matching →
 - $M[D]$, $M[D, C]$: contain 1 edge in  3 edges in 
 - $M[C]$: b -factor in $G[C]$
i.e., $\deg_M(u) = 2 \quad \forall u \in C$
 - $M[D, A]$: matches each vertex in A to distinct components in D



Characterization of max. square-free 2-matchings

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 - ✓ [Király '99]
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Summary & Future work

◆ Square-free 2-matchings in bipartite graphs: Well-solved

- Min-max theorems
 - Polyhedral description with dual integrality
 - Algorithms
 - Discrete convexity
 - Decomposition theorems [This talk]
-

◆ Lattice structure ??

◆ Subcubic graphs ??

◆ Application to TSP

cf. [Karp, Ravi '14]

9/7-approx. for the graph-TSP in cubic bipartite graphs

◆ Open problems:

- C_4 -free 2-matching
- Weighted C_3 -free 2-matching

Discrete convexity

[Kobayashi, Szabó, T. '12]

[Kobayashi '14]

- Barnette conjecture: 3-conn. planar cubic bipartite \rightarrow Hamiltonian ??

Min-max theorem [Király '99]

$G = (V, E)$: Bipartite

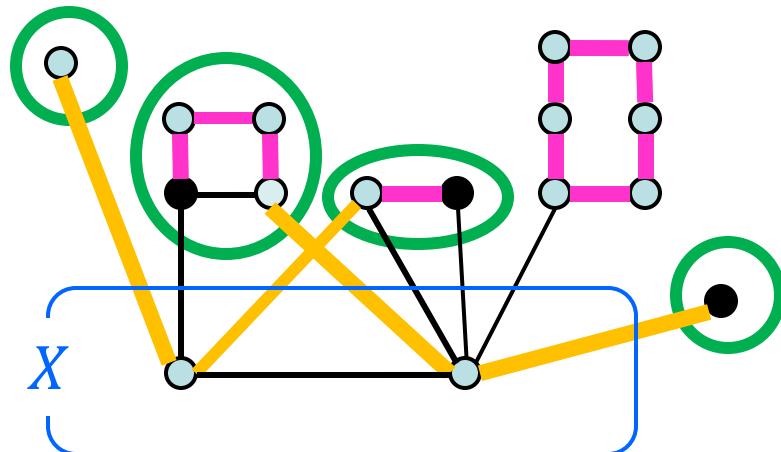
$$q(\bar{X}) = \#\{\textcircled{o}, \textcircled{o}-\textcircled{o}, \textcircled{o}-\textcircled{o}-\textcircled{o} \text{ in } G[\bar{X}]\}$$

Thm [Király '99]

$$\begin{aligned} & \max \{|M| : M \text{ is a } C_4\text{-free 2-matching}\} \\ &= \min \{|V| + |X| - q(\bar{X}) : X \subseteq V\} \end{aligned}$$

[$\max \leq \min$]

- $|M| = |M[\bar{X}]| + |M[X, \bar{X}]| + |M[X]|$
- $|M[X, \bar{X}]| + 2|M[X]| \leq 2|X|$
- $|M[\bar{X}]| \leq |\bar{X}| - q(\bar{X})$



Tutte-Berge formula (Nonbipartite matching)

$$\begin{aligned} & \max \{|M| : M \text{ is a matching}\} \\ &= \frac{1}{2} \min \{|V| + |X| - \text{odd}(\bar{X}) : X \subseteq V\} \end{aligned}$$

$$q(\bar{X}) = 4$$

Min-max theorem [Frank '03]

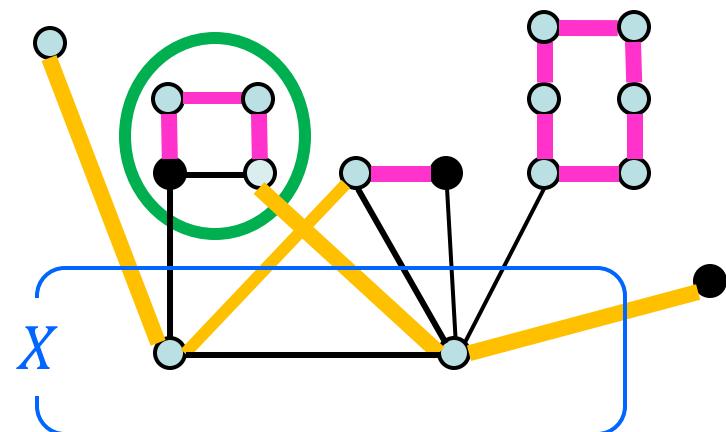
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Thm [Frank '03]

$$\begin{aligned} & \max \{ |M| : M \text{ is a } C_4\text{-free 2-matching} \} \\ &= \min \{ 2|X| + E[\bar{X}] - c(\bar{X}) : X \subseteq V \} \end{aligned}$$

[max ≤ min]

- $|M| = |M[\bar{X}]| + |M[X, \bar{X}]| + |M[X]|$
 - $|M[X, \bar{X}]| + 2|M[X]| \leq 2|X|$
 - $|M[\bar{X}]| \leq |E[\bar{X}]| - c(\bar{X})$



Thm (Bipartite 2-matching)

$$\max\{|M| : M \text{ is a 2-matching}\} = \min\{2|X| + |E[\bar{X}]| : X \subseteq V\}$$