Finding 2-Factors Closer to TSP Tours in Cubic Graphs

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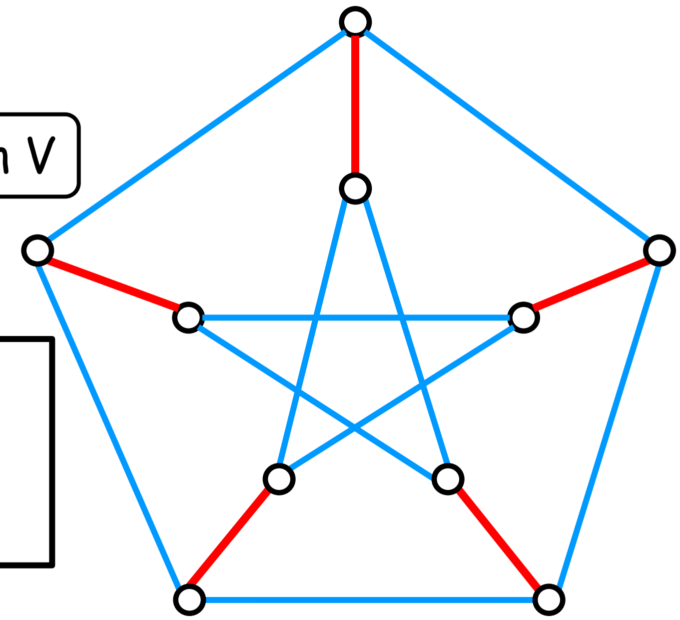
Petersen's Theorem

[1891]

G=(V,E): Bridgeless Cubic Graph

= 2-edge-connected

deg(v) = 3 for every v in V



Thm.

Every bridgeless cubic graph has a perfect matching

⇕

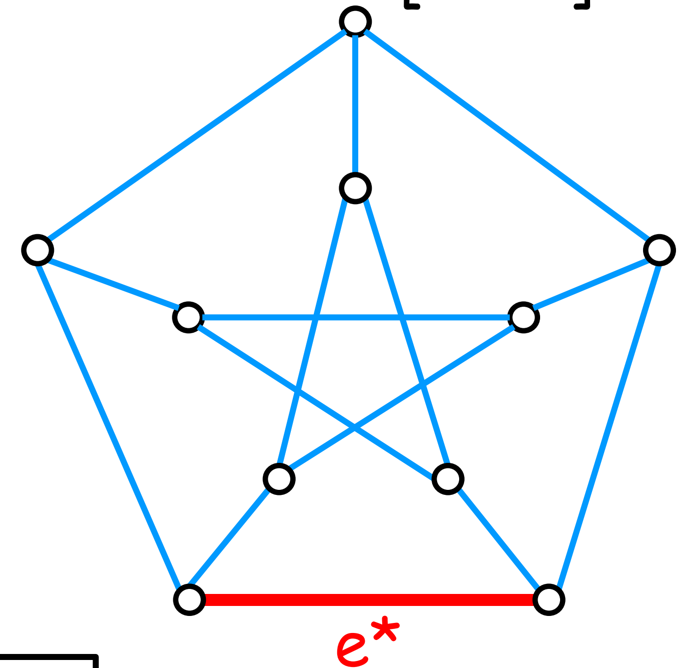
Every bridgeless cubic graph has a 2-factor

Schönberger's Theorem

[1935]

G=(V,E): Bridgeless Cubic Graph

e* in E



Thm.

G has a perfect matching including e*

⇕

G has a 2-factor excluding e*

- $O(n \log^4 n)$ algorithm
  [Biedl, Bose, Demaine, Lubiw 2001]

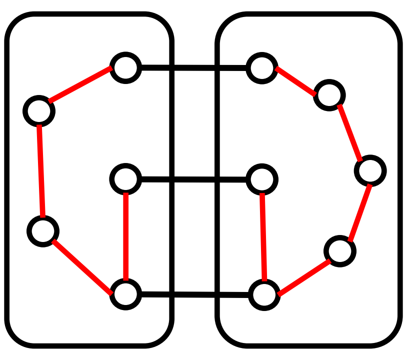
$n = |V|$

Kaiser & Škrekovski's Theorem

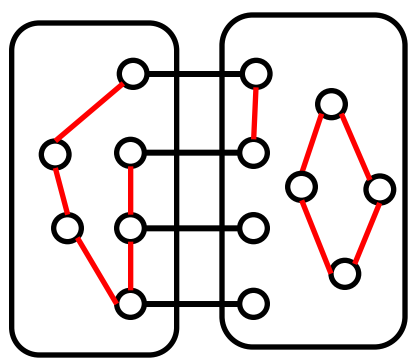
G=(V,E): Bridgeless Cubic [2008]

e* in E

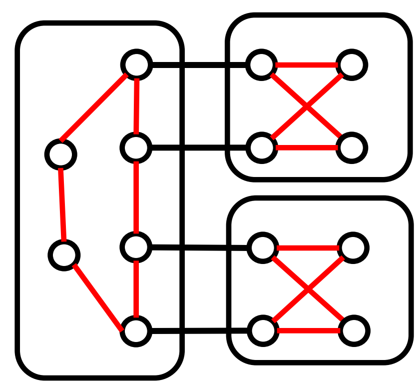
Thm. G has a 2-factor excluding e* and covering all 3- and 4-edge cuts



3-edge cut



4-edge cut

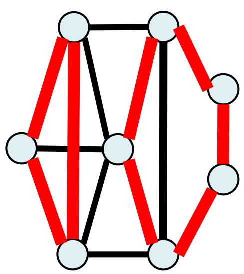


Not a 4-edge cut

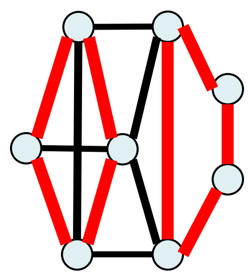
2-factors and TSP Tours

- TSP tour = 2-factor of one cycle of length n

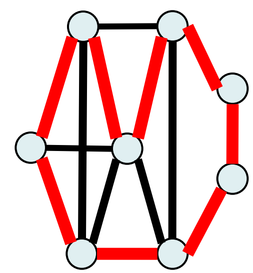
Relax ➔ 2-factor without cycles of length k or less :
$C_{\le k}$-free 2-factor (in simple graphs)



☹



✓ $C_{\le 3}$-free



✓ $C_{\le 4}$-free

k = n/2 ➔ TSP tour

Complexity of $C_{\leq k}$-free 2-factors

	Unweighted	Weighted
$k \geq 5$	NP-hard [Papadimitriou '80]	NP-hard
$k = 4$	(a) OPEN	(b) NP-hard [Vornberger '80]
$k = 3$	(c) P [Hartvigsen '84]	(d) OPEN
$k = 2$	P	P

- Subcubic graphs
  (a) : P [Bérczi & Végh '10]
  (c) : P [Bérczi & Végh '10, Hartvigsen & Li '11]
  (d) : P [Vornberger '80, Kobayashi '10, Hartvigsen & Li '13]
- Bipartite graphs
  (a) : P [Hartvigsen '06, Pap '07]
  (b) : NP-hard for general weight [Király 00]
  P if the weight hass a special property [Makai '07, T. '09]

2-factors Covering Cuts

- TSP tour = 2-factor covering all edge cuts

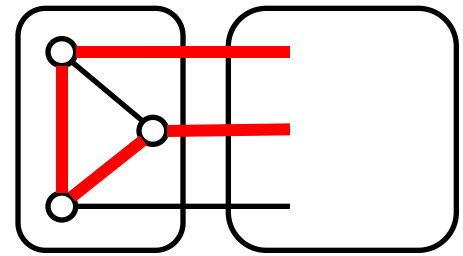
  ➔ 2-factor covering prescribed edge cuts

Relax

G: Cubic

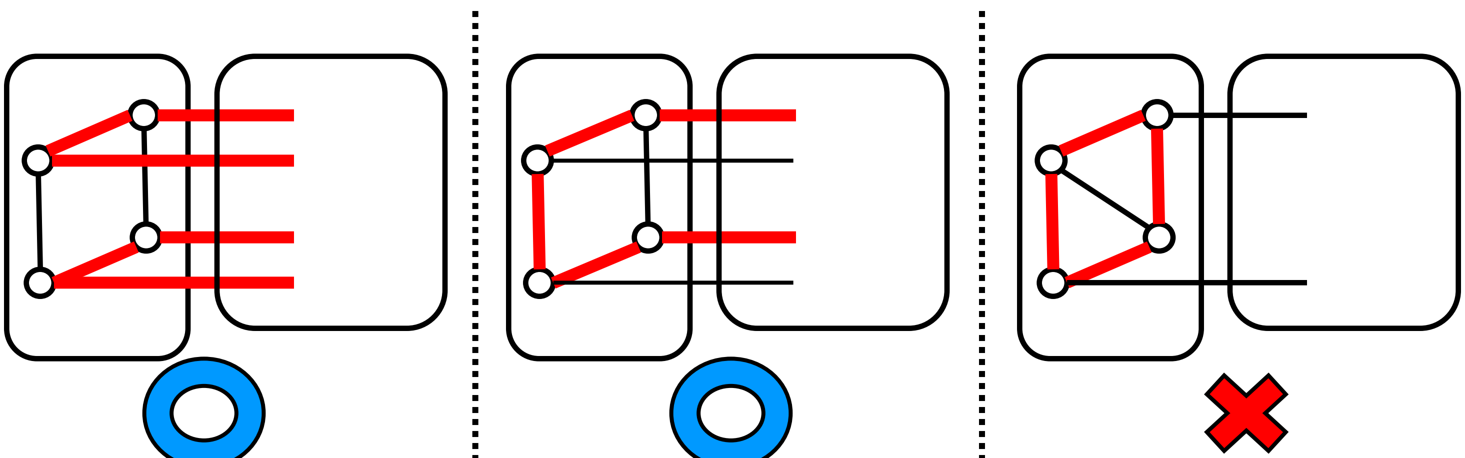
- 2-factor covering all 3-edge cuts

  ➔ $C_{\leq 3}$-free



G: 3-edge-connected cubic

- 2-factor covering all 3,4-edge cuts ➔ $C_{\leq 4}$-free



Our Results

(1) - An $O(n^3)$-algorithm for finding a *min.-weight* 2-factor covering all 3-edge cuts in bridgeless cubic graphs
- Polyhedral description

(2) An $O(n^3)$-algorithm for finding a 2-factor covering all 3-, 4-edge cuts in bridgeless cubic graphs

- Constructive proof for [Kaiser, Škrekovski 2008]

Application

(3) A 6/5-approx. algorithm for the minimum 2-edge-connected subgraph problem in 3-edge-connected cubic graphs

- Start with the 2-factor found by Algorithm (2)
- Previous ratio: 5/4 for 3-edge-connected cubic graphs [Huh 2004]

Contents

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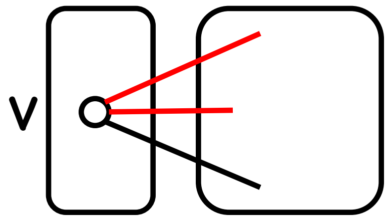
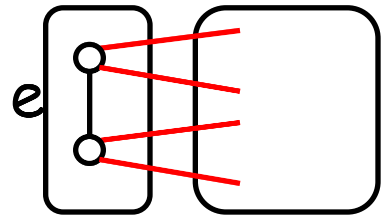
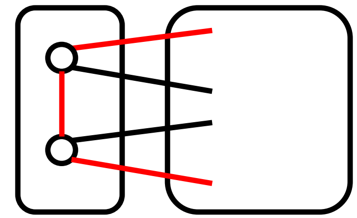
- Summary

Proper 3- and 4-Edge Cuts

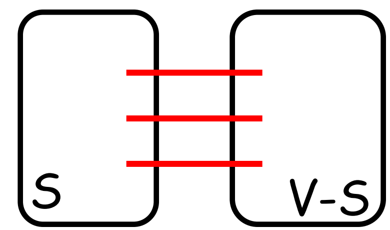
- 3- and 4-edge cuts covered by ***every*** 2-factor



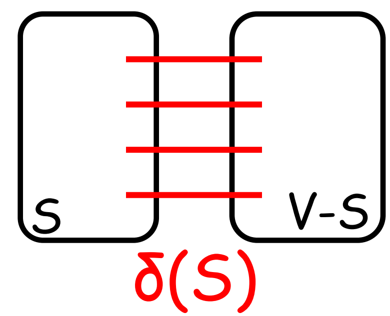




- A 3-edge cut $\delta(S)$ is proper
  $\Longleftrightarrow$ $2 \leq |S| \leq n - 2$



- A 4-edge cut $\delta(S)$ is proper
  $\Longleftrightarrow$ $3 \leq |S| \leq n - 3$



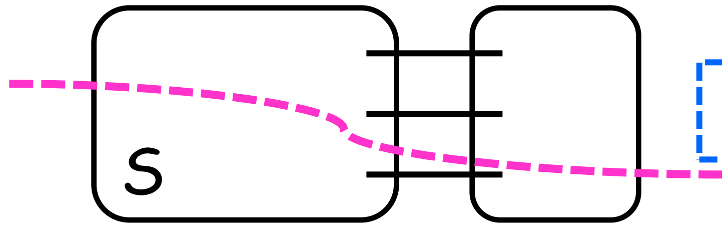
Goal

Find a 2-factor F satisfying:
- Covering all proper 3- and 4-edge cuts
- Excluding an edge e* in E

Covering 3-Edge Cuts

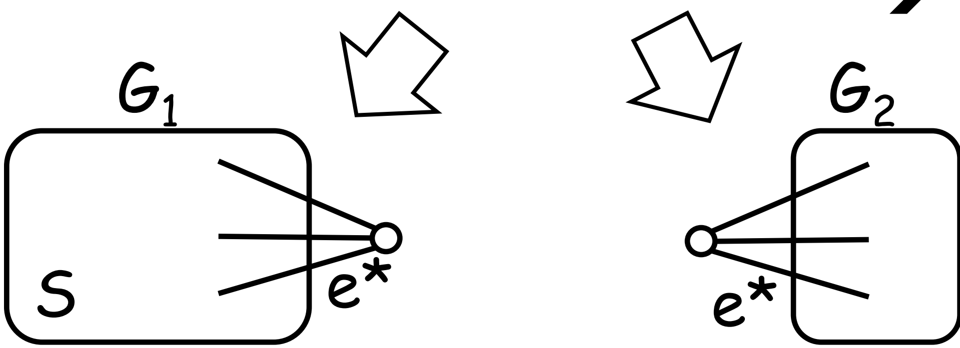
Gluing technique in [Cornuéjols, Naddef, Pulleyblank 85]

(1) Find a proper 3-edge cut $\delta(S)$



(2) Contract V - S, S

➔ Smaller bridgeless cubic graphs



(3) Recurse ➔ F_1

(4) Recurse ➔ F_2
(Exclude e*)

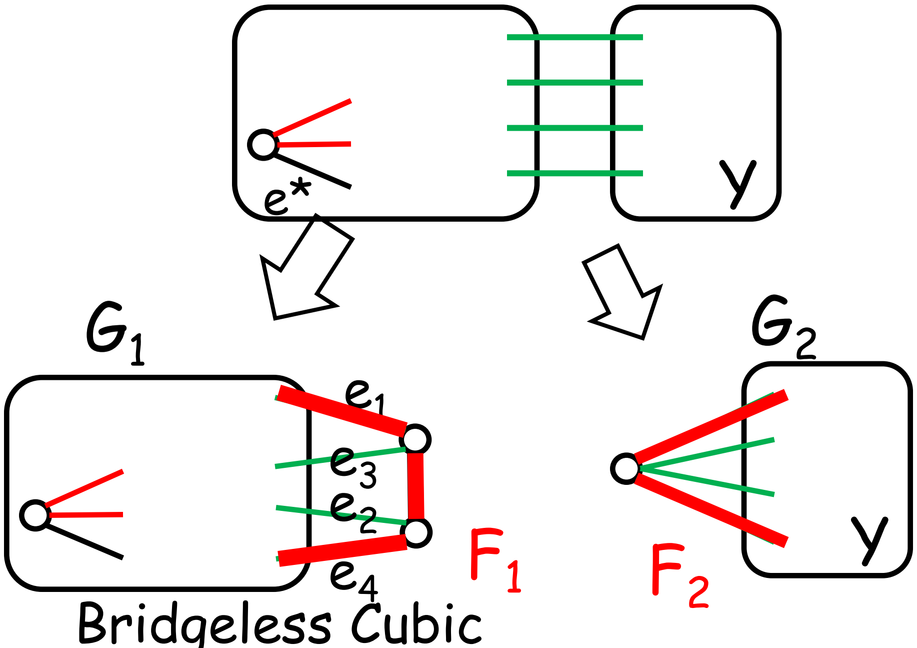
F covers all 3- & 4-edge cuts

(5) Return $F = F_1 + F_2$

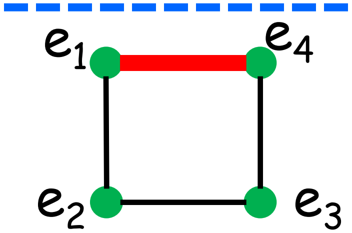
Covering 4-Edge Cuts

(1) Find a proper 4-edge cut $\delta(Y) = \{e_1,e_2,e_3,e_4\}$ (Y: minimal)

(2) Contract V - Y



(3) For any pair e_i,e_j, check if G_2 has a 2-factor including $\{e_i,e_j\}$

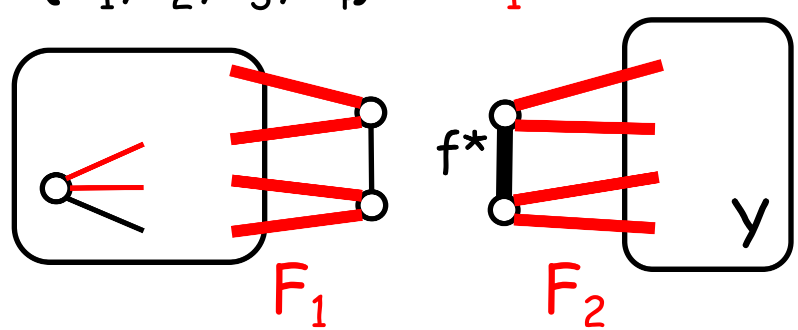


(4) Contract Y to v_Y, split v_Y according to (3)

(5) Recurse → F_1

(6) Return $F = F_1 + F_2$

- $\{e_1,e_2,e_3,e_4\}$ in F_1



Contents

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- Summary

The Minimum 2-Edge-Connected Subgraph Problem

Input: Graph $G = (V, E)$ Goal: 2-edge-connected subgraph (V, E') minimizing $\|E'\|$

- Hamilton cycle → Optimal solution
- n: lower bound

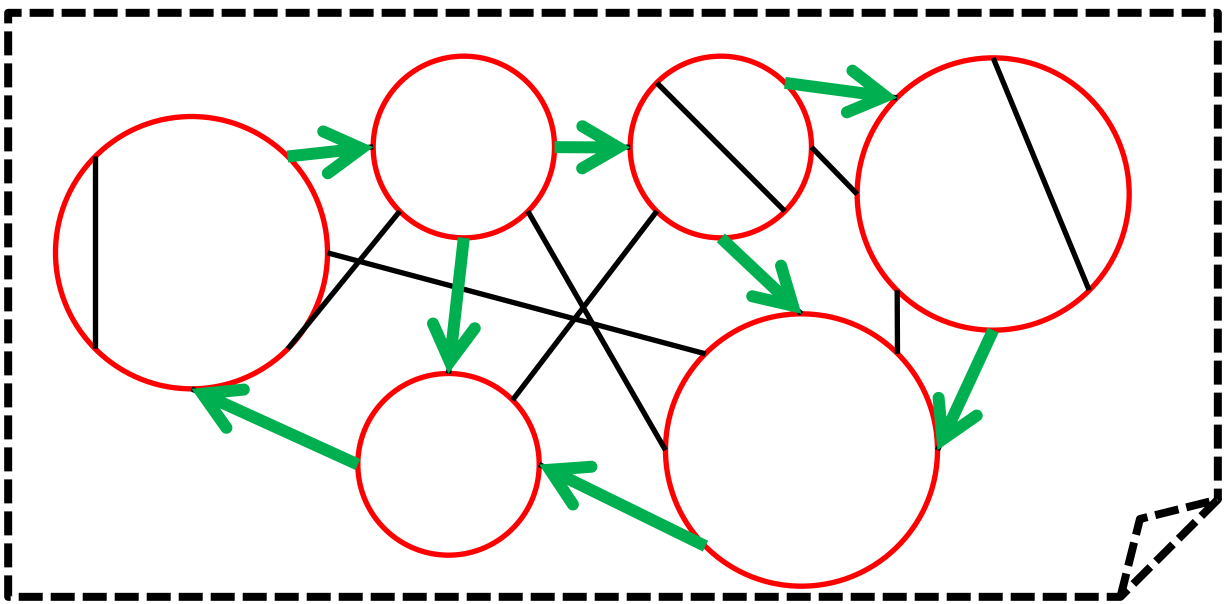
- General graphs

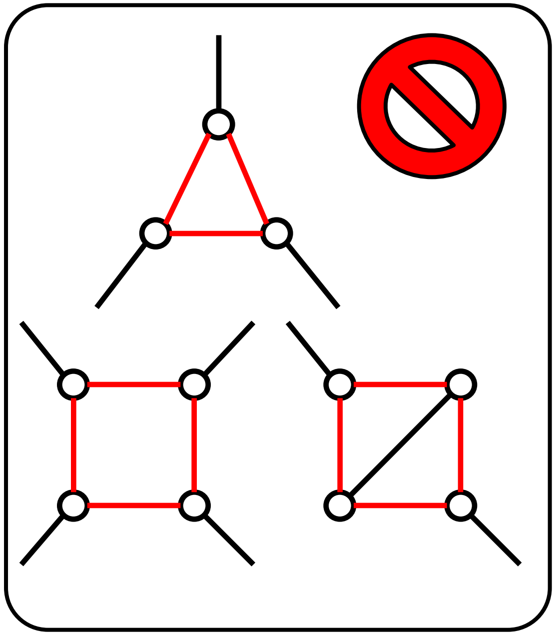
  Khuller, Vishkin ('94), Cheriyan, Sebő, Szigeti ('98)
  Vempala, Vetta ('00), Jothi, Raghavachari, Varadarajan ('03)
  Sebő, Vygen ('13): 4/3-approx.

- 3-edge-connected cubic graphs
  Huh ('04): 5/4-approx.
  This talk: 6/5-approx

Rough Idea

F: 2-factor covering all 3- and 4-edge cuts
➔ Cycles of length ≥ 5



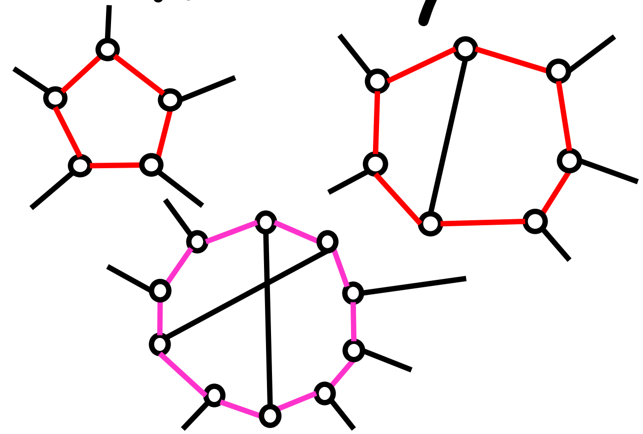


2 extra edges for each cycle
➔ 7/5-approx. ☹

Saving 1 Edge in a Small Cycle

Cycles in F :

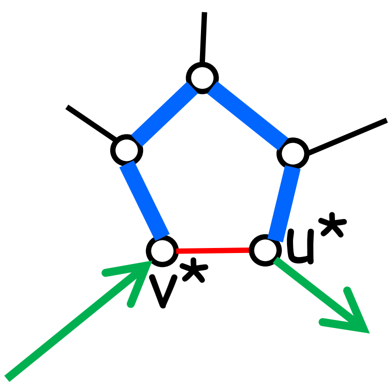
- Small cycle: Size 5--9
- Large cycle: Size ≥ 10

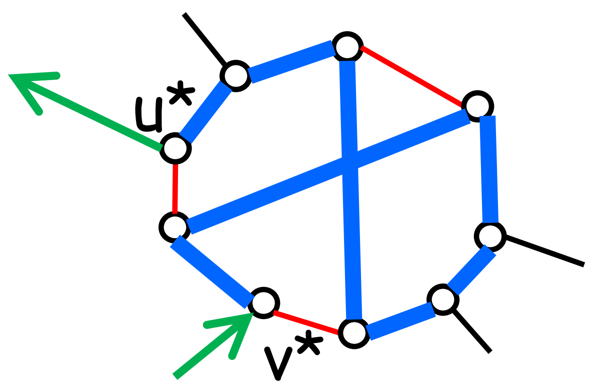


Lemma

C: Small cycle in F
We have reached at v* in V(C)

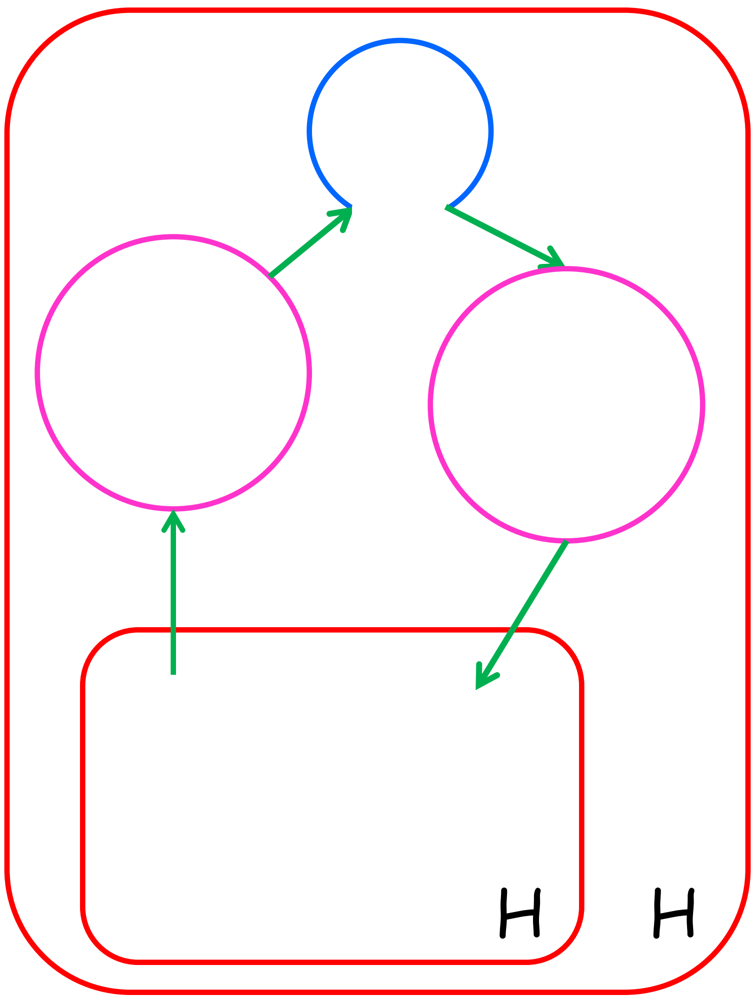
➔ G[V(C)] has a Hamilton path from v* to u*, and we can leave for another cycle from u*



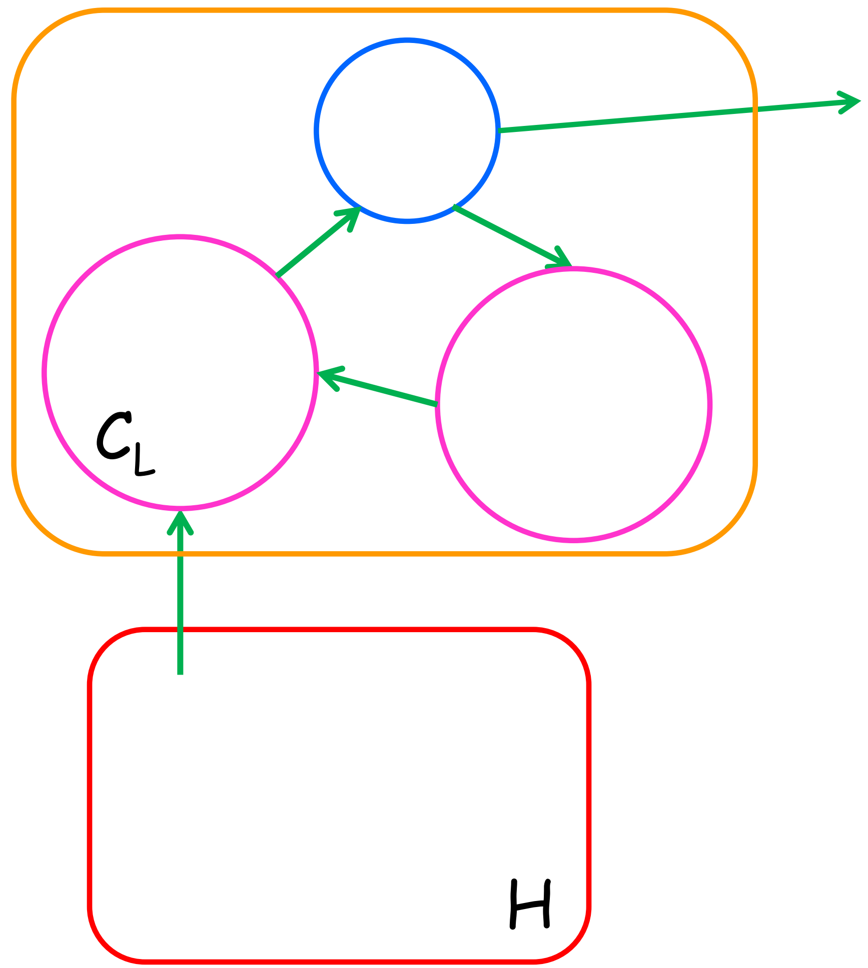


Algorithm Sketch

- Back to H ➔ Update H

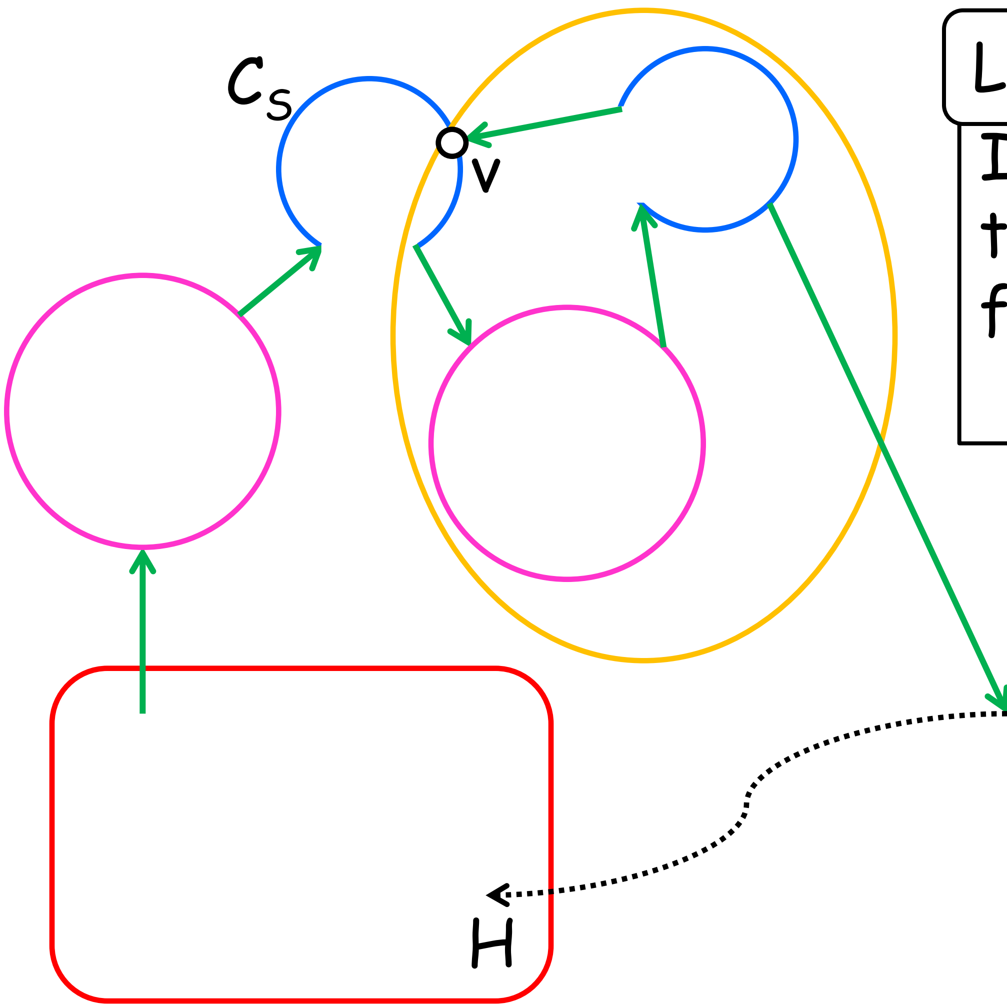


- Back to a large cycle C_L ➔ Compound C_L--C_L

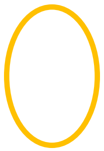


Algorithm Sketch

- Back to a small cycle C_S (at v in V(C_S)) ➔ Compound v--v



Lemma

If G is 3-edge-connected, there exists an edge from  to another cycle.

Approximation Ratio

Thm.

$|E(H)| \leq 6n/5 - 1$

(Pf.) x = # small cycles in the initial 2-factor F

y = # large cycles in the initial 2-factor F

2 extra edges for each cycle

Save 1 edge for each small cycle

$$|E(H)| \leq n + 2(x + y - 1) - (x - 1)$$

$$= n + x + 2y - 1$$

$5x + 10y \leq n$

$$\leq 6n/5 - 1$$

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Summary

- For bridgeless cubic graphs:
  - A 2-factor covering all 3- and 4-edge cuts: Algorithm
  - A min-weight 2-factor covering all 3-edge cuts:
    - ✓ Algorithm
    - ✓ Polyhedral description
- For 3-edge-connected cubic graphs
  - 6/5-approx. algorithm for the min. 2-edge-connected subgraph problem

Open Problems

Min-weight 2-factor covering all 3- and 4-edge cuts in bridgeless cubic graphs

6/5-approx. algorithm for the min. 2-edge-connected subgraph problem in ***bridgeless*** cubic graphs

2-Factors Covering 3-Edge Cuts [Weighted]

2-factor polytope [Edmonds 1965]

$x(\delta(v)) = 2$	v in V
$x(Y) - x(\delta(S) - Y) \le \lvert Y\rvert - 1$	$S \subset V$, $Y \subseteq \delta(S)$, Y: matching, $\lvert Y\rvert$: odd
$0 \le x(e) \le 1$	e in E

Additional constraint

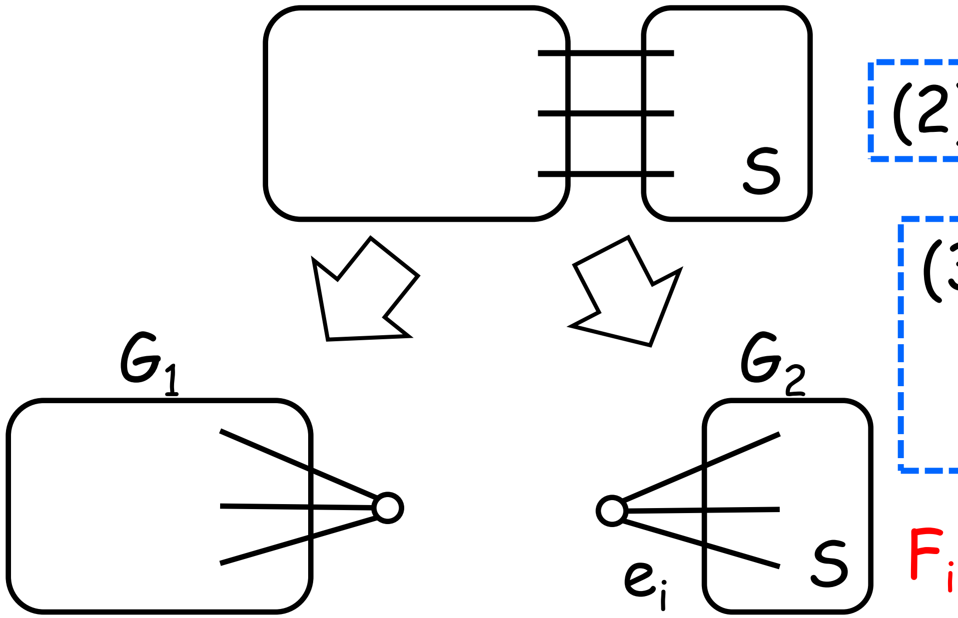
$x(\delta(S)) = 2$	$S \subset V$, $\delta(S)$ is a 3-edge cut

<u>Thm.</u>
The above constraints determine the polytope of the 2-factors covering all 3-edge cuts

Algorithm Sketch

Gluing technique in [Cornuéjols, Naddef, Pulleyblank 85]

(1) Find a proper 3-edge cut $\delta(S) = \{e_1, e_2, e_3\}$ (S: minimal)



(2) Contract V - S, S

(3) In G_2, find a min. weight 2-factor F_i excluding e_i (i=1,2,3)

- $L_i = w(F_i \cap E[S])$

(3) In G_1, add extra weight x_i for e_i, where $x_1 + x_2 = L_3$, $x_2 + x_3 = L_1$, $x_3 + x_1 = L_2$

(4) Recurse in G_1